Multi-Scale Variance Stabilizing Transform for Multi-Dimensional Poisson Count Image Restoration

B. Zhang, M. J. Fadili and J.-L. Starck

UAIQ URA CNRS 2582, Institut Pasteur Paris, France

GREYC UMR CNRS 6072 Caen, France

DAPNIA/SEDI-SAP CEA-Saclay, Gif-sur-Yvette, France

ICASSP 06

Plan

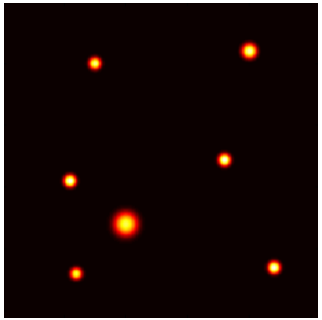
- Introduction
- Variance Stabilizing Transform (VST)
- Application to multi-scale transforms (MS-VST)
- Results
- Conclusion

Introduction – Context

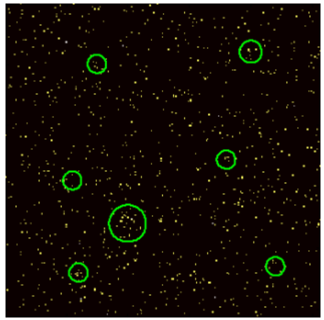
- GLAST project (The Gamma Ray Large Area Space Telescope 2007, Stanford Univ.)
- Images of photons emitted by gamma sources

Introduction – Data

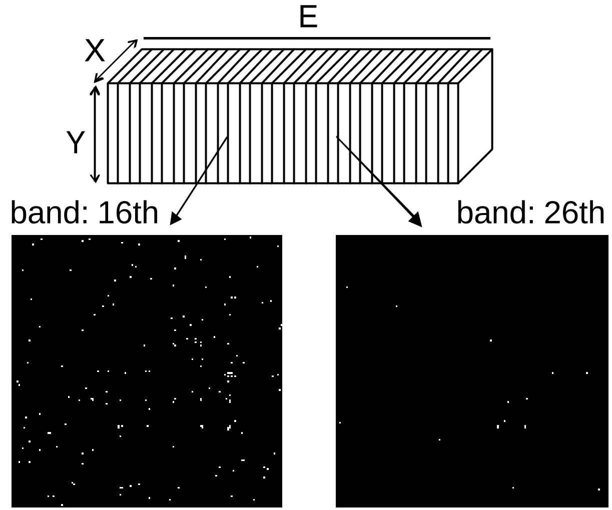
underlying sources



counts + src positions

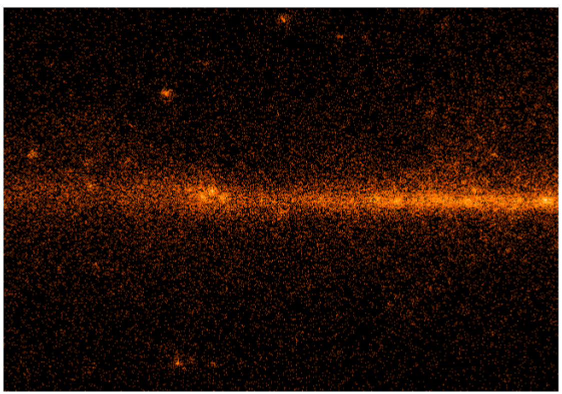


very low source intensity (~10^{-1} or even lower)

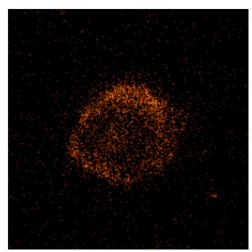


multi-spectral images

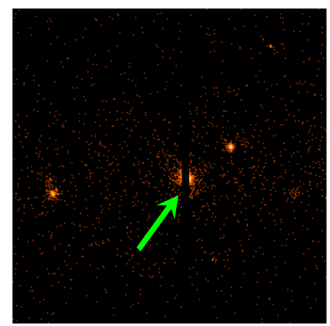
linelike structure



curvilinear structure



non-isotropic sources

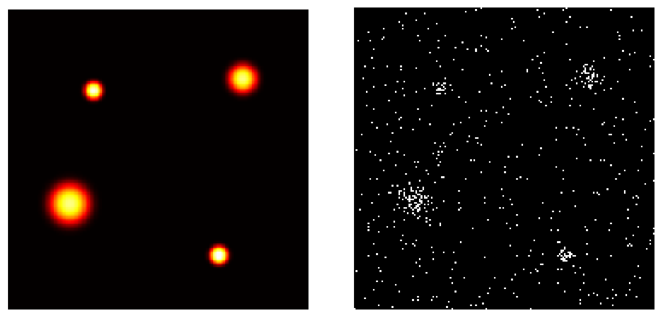


defective camera channels

Introduction – Image Model and Goals

- We observe a discrete image of counts $X = (X_i)_{i \in \mathbb{Z}^d}$ where $X_i \sim \mathcal{P}(\lambda_i)$ are independent Poisson variables; $\Lambda = (\lambda_i)_{i \in \mathbb{Z}^d}$ is thus the underlying intensity image
- Example

intensity image counts (observed)



- Goals
  - Statistical intensity estimation in a sparse-representation domain $\hat{\Lambda} = \mathcal{D}X$
  - Efficient for very low-count settings
  - Shape-adaptive estimation (isotropic, linelike, curvilinear, etc.)
  - Missing data restoration
  - Capable of dealing with multi-spectral images
  - Photometry preservation

Generalize the Anscombe Transform: VST for a Filtered Poisson Process

- Anscombe transform (rapid but efficient only for high-count settings)

$$X \sim \mathcal{P}(\lambda), \quad 2\sqrt{X+3/8} \sim \mathcal{AN}(2\sqrt{\lambda}, 1), \quad \lambda \to \infty$$

- Anscombe transform: stabilization after filtering with $h = \delta$
- Generalize the VST for a non-trivial filter h

$$Y_n = \sum_i h[i] X_{n-i}, \; X_i \sim \mathcal{P}(\lambda)$$

  - h acts as an “averaging” kernel (more generally a low-pass filter)
  - The SNR is enhanced after filtering
  - ✓ Stabilization is more efficient in low-count settings
  - ✓ The shape can be preserved by designing matched filter
- The general form of the VST (some heuristics)

$$T(Y) \approx T(\mu) + T'(\mu)(Y-\mu) \;\Rightarrow\; 1 \equiv Var[T(Y)] \approx T'(\mu)^2 Var(Y) \qquad \mu = \tau_1 \lambda$$

$$T'(u) = \sqrt{\frac{\tau_1}{\tau_2}} u^{-\frac{1}{2}} \;\Rightarrow\; T(u) = b\sqrt{u} \qquad Var(Y) = \tau_2 \lambda$$

$$\tau_k = \sum_i h[i]^k$$

square root as VST

VST for a Filtered Poisson Process

Theorem 1 *Define* $T(Y) = b\sqrt{Y + c}$. *Then, we have the following results:*

(i) $T(Y) - b\sqrt{\tau_1 \lambda} \xrightarrow{\mathcal{D}} \mathcal{N}(0, b^2 \frac{\tau_2}{4\tau_1})$ *as* $\lambda \to \infty$, *for* $c > 0$, $\tau_1 > 0$ *and* $\tau_2 > 0$;

(ii) *The asymptotic expansions of the mean and variance of* $T(Y)/b$ *are given by:*

$$\mathbb{E}(T(Y)/b) = \sqrt{\lambda \tau_1} + \frac{4c\tau_1 - \tau_2}{8\tau_1^{3/2}} \lambda^{-1/2} + O(\lambda^{-3/2})$$

$$Var(T(Y)/b) = \frac{\tau_2}{4\tau_1} + \underbrace{\left(\frac{7\tau_2^2}{32\tau_1^3} - \frac{2\tau_2 c + \tau_3}{8\tau_1^2} \right)}\lambda^{-1} + \left(\frac{c^2\tau_2 + c\tau_3}{4\tau_1^3} + \frac{5\tau_4}{64\tau_1^3} - \frac{17\tau_2\tau_3 + 21c\tau_2^2}{32\tau_1^4} + \frac{285\tau_2^3}{1024\tau_1^5} \right) \lambda^{-2} + O(\lambda^{-5/2})$$

(iii) *For the VST to be second order accurate and* $T(Y)$ *to have asymptotic unit variance,* b *and* c *should satisfy:*

$$b = 2\sqrt{\frac{\tau_1}{\tau_2}} \qquad c = \frac{7\tau_2}{8\tau_1} - \frac{\tau_3}{2\tau_2}$$

$h = \delta \quad \tau_i = 1$

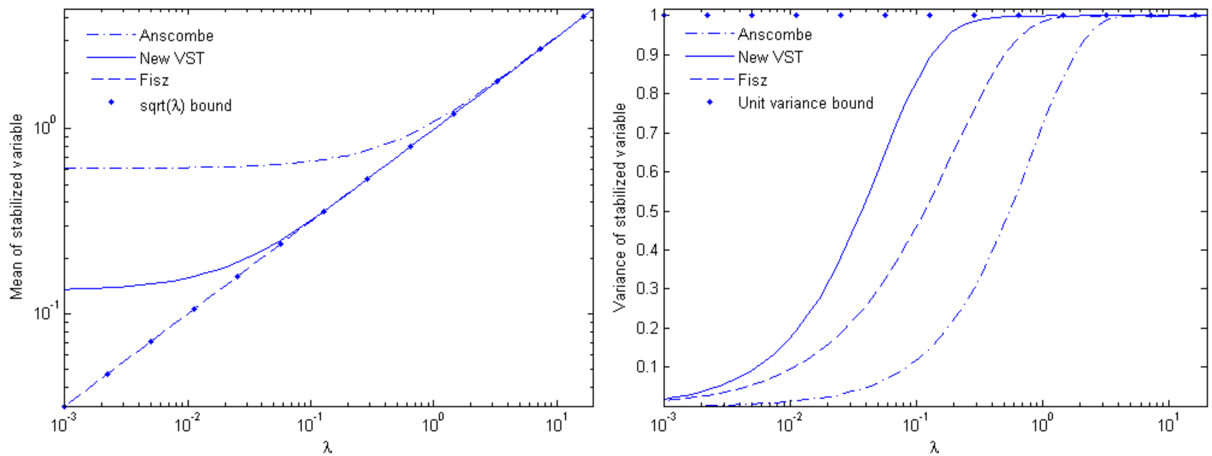
$c = \frac{3}{8}$ (Anscombe)

Asymptotic Results

- The variance is constant up to a second order residual term
- For appropriately chosen h, the convergence toward the asymptotic behavior can be much rapid for the new VST than Anscombe
- Example

next-higher-order coefficients in the asymptotic expansions

Filter	C_E	C_{Var}
$\delta = [1]$ (Anscombe)	6.25×10^{-2}	-1.1680
B_3-Spline $= [1\ 4\ 6\ 4\ 1]/16$	8.22×10^{-3}	-9.22×10^{-2}



→ The Poisson process can be reasonably considered stabilized for $\lambda \gtrsim 5$ using Anscombe, for $\lambda \gtrsim 1$ using Fisz and for $\lambda \gtrsim 0.1$ using the new VST+ h_{B_3}

Application to Wavelets: Multi-Scale VST (MS-VST)
Isotropic Source Restoration

- The UWT $\mathcal{W}$ uses the filter bank (h, g) to decompose an 1D signal into a set of coefficients

$$a_{j+1} = a_j \star \bar{h}_j, \quad w_{j+1} = a_j \star \bar{g}_j$$

$$\left(\bar{h}[n] = h[-n],\ \bar{g}[n] = g[-n]\right)$$

- The original signal is reconstructed by the filter bank $(\tilde{h}, \tilde{g})$

$$a_j = a_{j+1} \star \tilde{h}_j + w_{j+1} \star \tilde{g}_j$$

- The filter bank needs only verify the exact reconstruction condition

$$\hat{h}^* \hat{\tilde{h}} + \hat{g}^* \hat{\tilde{g}} = 1$$

Application to Wavelets: Multi-Scale VST (MS-VST)
Isotropic Source Restoration

- The filter bank $\left(h = h_{B_n}, g = \delta - h, \tilde{h} = \delta, \tilde{g} = \delta\right)$ is well adapted to isotropic sources and is extensively used in astronomy, biomedical imaging, etc.
- We can apply a Multi-Scale VST (MS-VST) for this filter bank as follows:

$$\begin{aligned} a_j &= h_j \star a_{j-1} \\ w_j &= a_{j-1} - a_j \end{aligned} \quad \Longrightarrow \quad \begin{aligned} a_j &= h_j \star a_{j-1} \\ w_j &= T_{j-1} a_{j-1} - T_j a_j \end{aligned}$$

Theorem 2 *Suppose* h_{B_n} *to be normalized such that* $\|h_{B_n}\|_1 = 1$. *Consider the VST:* $T_j(a_j) = \left[a_j + c^{(j)}\right]^{\frac{1}{2}}$ *where* $c^{(j)}$ *is derived from with the filter* $h^{(j)} = h_1 \star \cdots \star h_j$. *Then, for any given scale* j, *as* $\lambda \to +\infty$,

$$w_j \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \quad \frac{\|h^{(j-1)}\|_2^2}{4} + \frac{\|h^{(j)}\|_2^2}{4} - \frac{<h^{(j-1)}, h^{(j)}>}{2}\right)$$

→ Apply your favorite denoising method for Gaussian noise

- All the results hold for any dimensional case

MS-VST + Wavelet: Algorithm Sketch

1. Let $a_0 = \mathbf{X}$, for a given filter h
2. **for** j = 1 **to** J
3. $a_j = h_j \star a_{j-1}$
4. $w_j = T_{j-1} a_{j-1} - T_j a_j$
5. Denoise (typically thresholding) w_j assuming a zero-mean Gaussian noise, to obtain $\hat{w}_j$
6. **end for**
7. Reconstruct

$$\widehat{T_0 a_0} = T_J a_J + \sum_{j=1}^{J} \hat{w}_j$$

$$\begin{aligned} \hat{a}_0 &= Var(T_0 a_0) + \widehat{T_0 a_0}^2 - c^{(0)} \\ &= \widehat{T_0 a_0}^2 + \left[\frac{1}{b^{(0)}}\right]^2 - c^{(0)} \end{aligned}$$

A More General Scheme

- For an arbitrary band-pass filter g

$$
\begin{aligned}
a_j &= h_j \star a_{j-1} \\
w_j &= g_j \star T_{j-1} a_{j-1}
\end{aligned}
$$

Theorem 3 *We have* $w_j[l] \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2_{j,l})$ *as* $\lambda \to +\infty$, *where*

$$
\begin{aligned}
\sigma^2_{j,l} &= \frac{1}{\tau_2^{(j-1)}} \sum_{m,n} g_j[l-m] g_j[l-n] \sum_k h^{(j-1)}[k] h^{(j-1)}[n-m+k] \\
&= \frac{1}{\tau_2^{(j-1)}} \sum_{m,n} g_j[m] g_j[n] \sum_k h^{(j-1)}[k] h^{(j-1)}[n-m+k]
\end{aligned}
$$

- Detection: use the same schema as before
- Estimation and reconstruction: Optimize the sparseness of the representation with constraints (using Hybrid Steepest Descent iteration)

Extensions

- **Multi-Spectral data** MS-VST + [2D+1D] Wavelet (number of scales different in each direction)
- **Shape adaption** MS-VST + Ridgelet and MS-VST + Curvelet
- **Defective channels** MS-VST+ x-Let + EM-inpainting

Green: solved
Blue: currently working on

Results – Isotropic Sources Restoration (MS-VST + Wavelet)

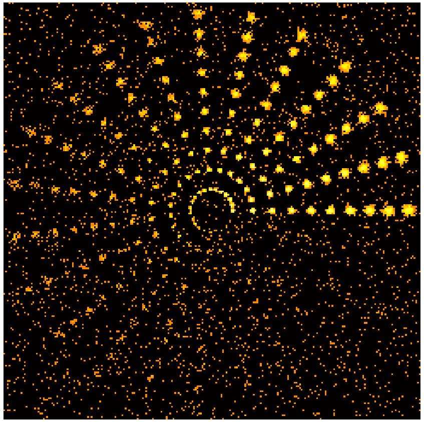
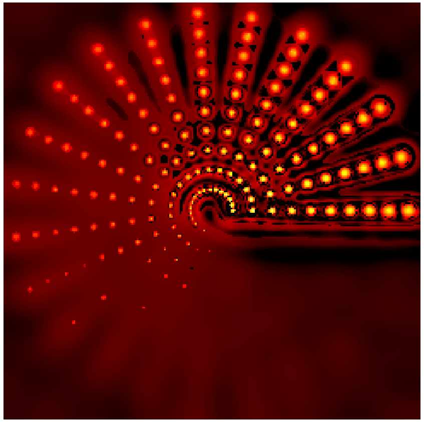
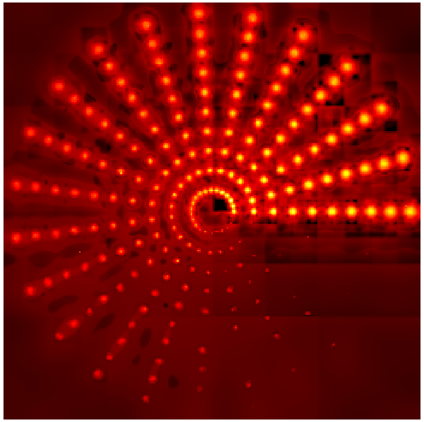
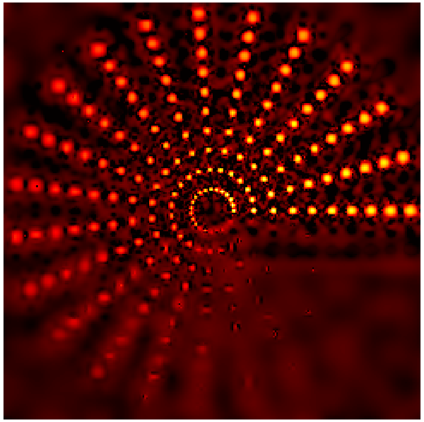
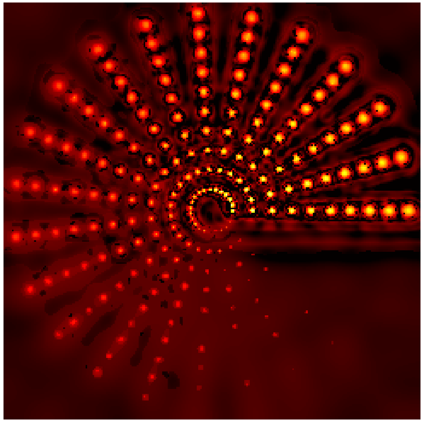


image of counts


Anscombe


Fisz + 25 Cyc. Spin.


Bi-Haar (direct thr.)


MS-VST

Thresh. Level = 4s

Results – Isotropic Sources Restoration (MS-VST + Wavelet)

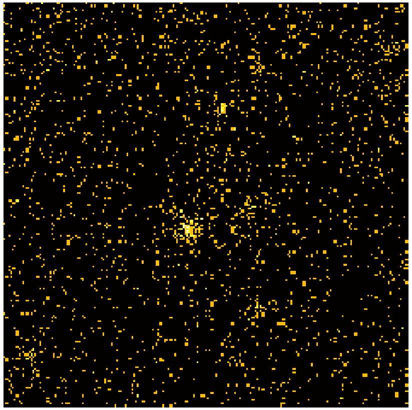
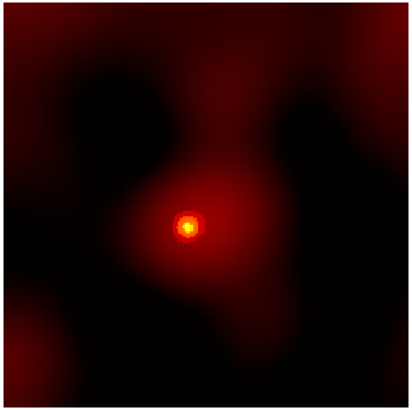
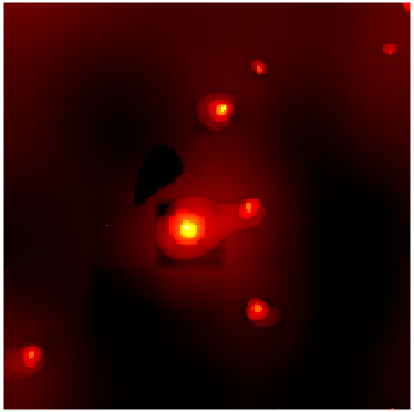
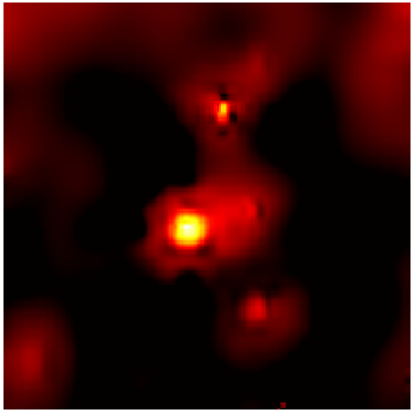
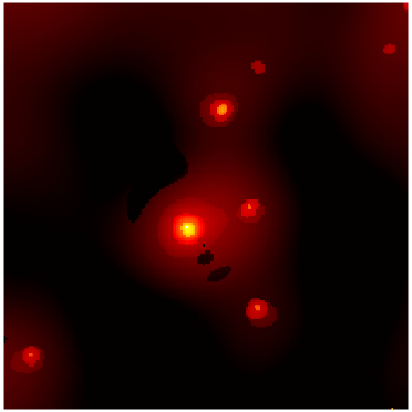


image of counts


Anscombe

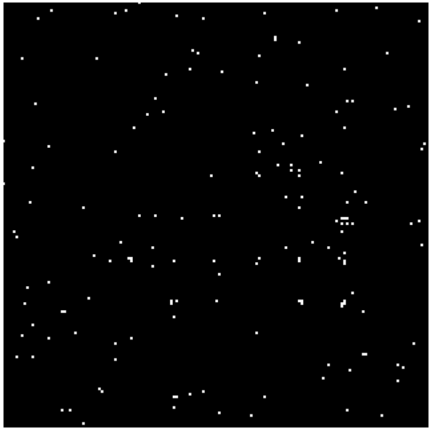

Fisz + 25 Cyc. Spin.

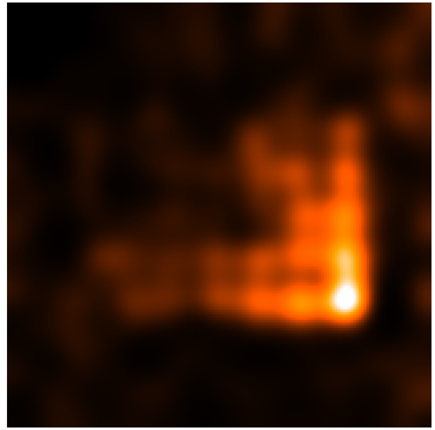

Bi-Haar (direct thr.)

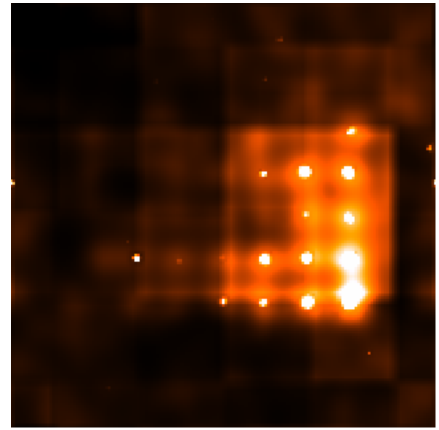

MS-VST

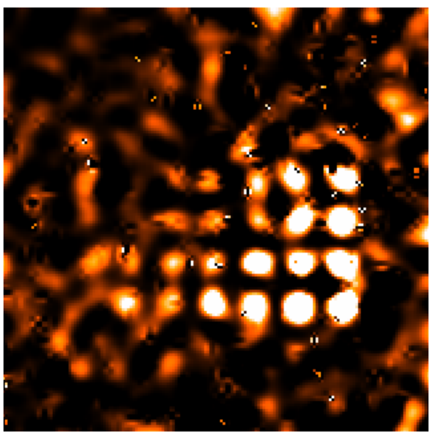
Thresh. Level = 4s

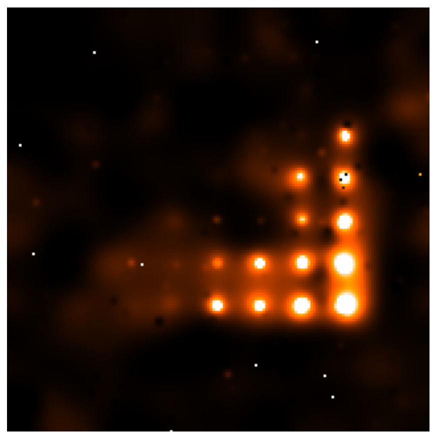
Results – Multi-Spectral Image Restoration (MS-VST + [2D+1D] Wavelet)


16th band


Anscombe


Fisz + 64 Cyc. Spin.

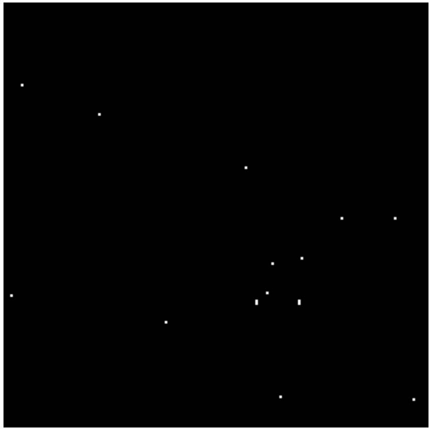

Bi-Haar (direct thr.)


MS-VST

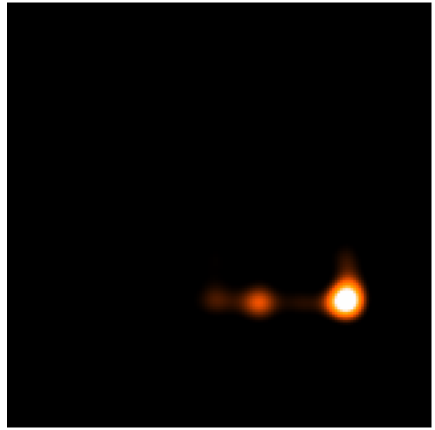
A 35-source grid:
5 lines x 7 columns

Thresh. Level = 4s

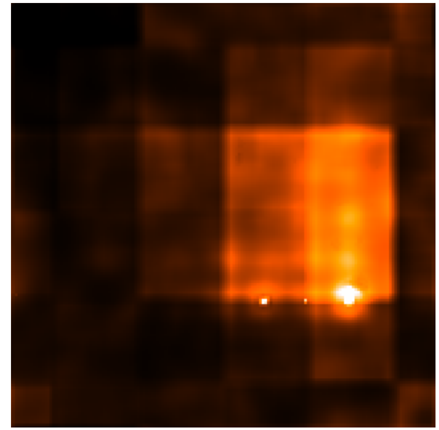
Results – Multi-Spectral Image Restoration (MS-VST + [2D+1D] Wavelet)



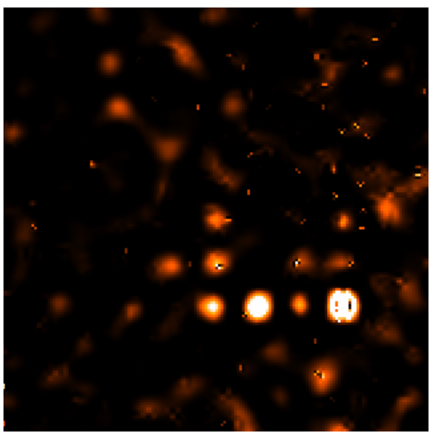
26th band



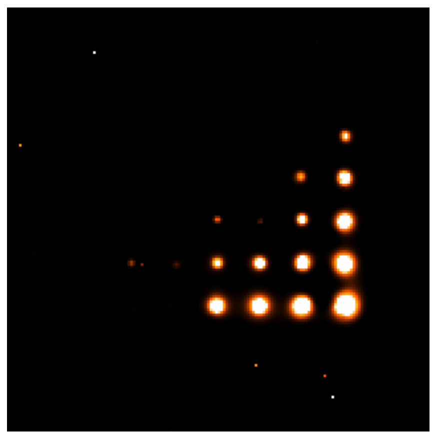
Anscombe



Fisz + 64 Cyc. Spin.



Bi-Haar (direct thr.)



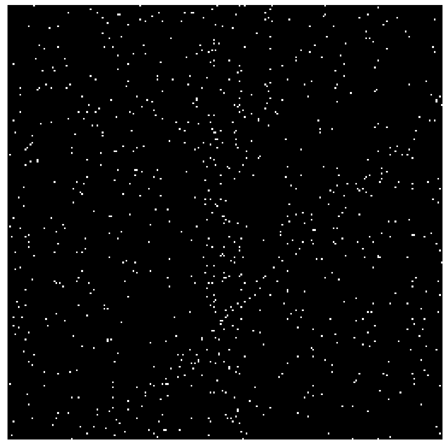
MS-VST

A 35-source grid:
5 lines x 7 columns

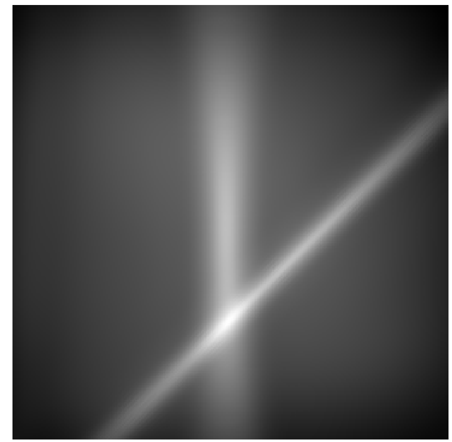
Thresh. Level = 4s

Preliminary Results – Image Restoration with Line-Like Sources (MS-VST + Ridgelet)

underlying intensity image



simulated image of counts



restored image
from the left image of counts

Max Intensity
background = 0.01
vertical bar = 0.03
inclined bar = 0.04

Preliminary Results –Restoration with Missing Data (MS-VST + Wavelet + Inpainting)

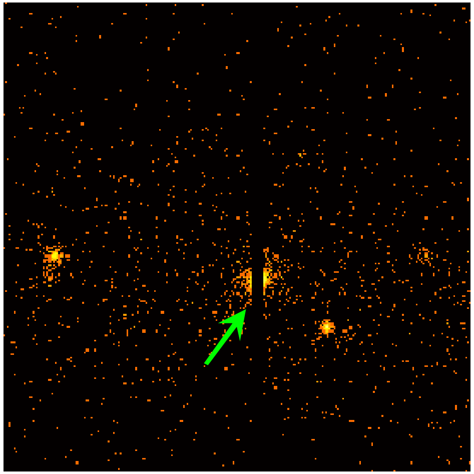
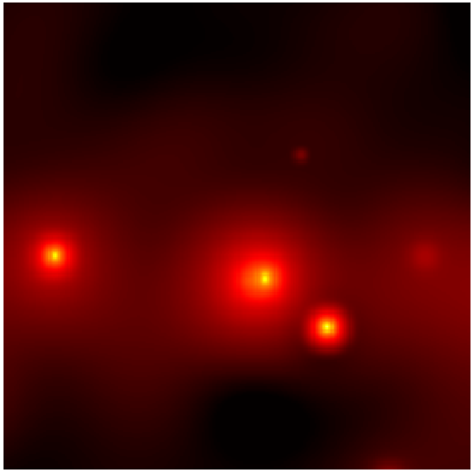


image of counts
with defective channels



restored image
from the left image
with missing observations

Conclusion and Perspectives

- VST for low-pass filtered Poisson process
  - Efficient for (very) low-count settings
- MS-VST: the VST can be naturally coupled with most multi-scale transforms (e.g. x-let)
  - Multi-spectral data
  - Shape adaptation
  - Regularity preservation
- MS-VST is fast and easy to implement in any dimension
- Currently working issus and perspectives: MS-VST with other transforms, with inpainting, with deconvolution, etc.

Acknowledgements

- **S. W. Digel and J. Chiang** (Stanford Univ.)
- **J.-C. Olivo-Marin** (Institut Pasteur)