Total generalized variation: From regularization theory to applications in imaging

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Outline

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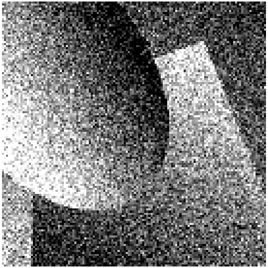
The total variation model in imaging



f

Problem:

Reconstruct image u from

- (blurred) noisy data
- noisy indirect measurements, etc.

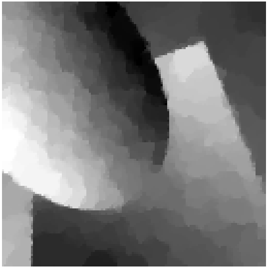
Widely used approach:

$$\min_{u \in \mathrm{BV}(\Omega)} \frac{\|Ku - f\|_2^2}{2} + \alpha \, \mathrm{TV}(u)$$

Total variation TV:

- Convex energy
- Allows for discontinuities
- Enforces “sparse” gradient

The total variation model in imaging



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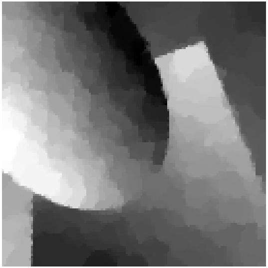

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- Unaware of higher-order smoothness
- Texture is not captured

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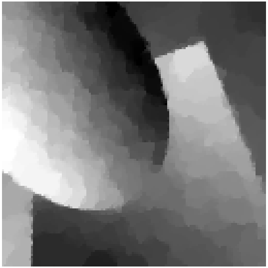

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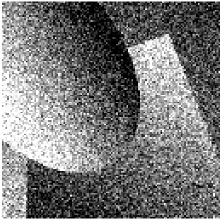

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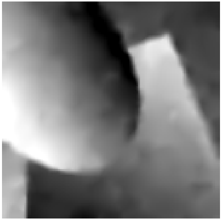

Higher-order TV: $\Phi(u) = \int_\Omega \mathrm{d}|\nabla^2 u|$

[Lysaker/Lundervold/Tai '03]
[Hinterberger/Scherzer '04]

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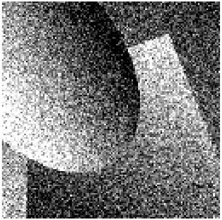

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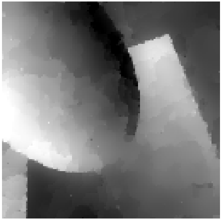

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TV-TV^2 **infimal convolution:**

$$\Phi(u) = \min_{u=u_1+u_2} \int_\Omega \mathrm{d}|\nabla u_1| + \beta \int_\Omega \mathrm{d}|\nabla^2 u_2|$$

[Chambolle/Lions '97]

- Models piecewise smooth images
- Staircase effect dominates solutions

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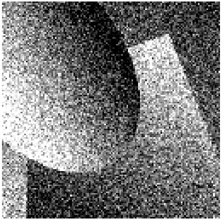

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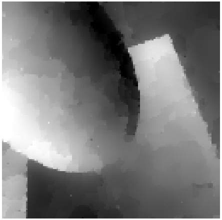

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$\rightsquigarrow$ **Different approach is needed**

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1. Introduction
2. Total Generalized Variation
   - Existence and stability for second order
   - Regularization theory for general orders
   - Optimization algorithms
3. Applications
   - Compressive imaging
   - JPEG(2000) decompression and zooming
   - Quantitative susceptibility mapping
   - Dual energy CT denoising
4. Summary

Total Generalized Variation

Joint work with Karl Kunisch and Thomas Pock

Definition: Total Generalized Variation

$$\mathrm{TGV}_\alpha^k(u) = \sup \Big\{ \int_\Omega u \,\mathrm{div}^k v \,\mathrm{d}x \;\Big|\; v \in \mathcal{C}_c^k(\Omega, \mathrm{Sym}^k(\mathbb{R}^d)), \\ \|\mathrm{div}^l v\|_\infty \leq \alpha_l, \, l = 0, \ldots, k-1 \Big\}$$

- $\alpha = (\alpha_0, \ldots, \alpha_{k-1}) > 0$ weights

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Idea:

- Incorporate information from $\nabla u, \ldots, \nabla^k u$
- Formal observation:

$$\int_\Omega |\nabla^k u| \qquad dx = \sup \Big\{ \int_\Omega u \,\mathrm{div}^k v \, dx \ \Big| \ v \in \mathcal{C}_c^k(\Omega, \mathrm{Sym}^k(\mathbb{R}^d)), \ \|v\|_\infty \le 1 \qquad \Big\}$$

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Basic properties: *[B./Kunisch/Pock '10]*

- TGV_α^k is proper, convex, lower semi-continuous
- TGV_α^k is translation and rotation invariant
- $\mathrm{TGV}_\alpha^k + \|\cdot\|_1$ gives Banach space $\mathrm{BGV}_\alpha^k(\Omega)$
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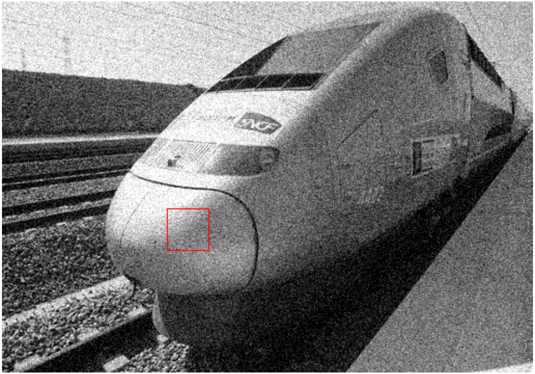
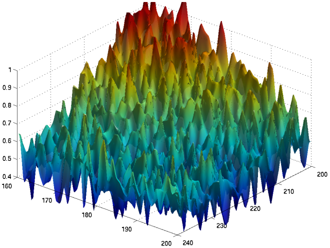
Application: Denoising

Solve: $$\min_{u \in L^2(\Omega)} \frac{\|u - f\|^2}{2} + \mathrm{TGV}_\alpha^k(u)$$





Noisy image

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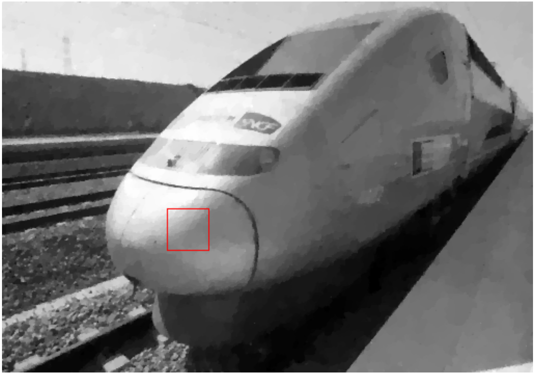

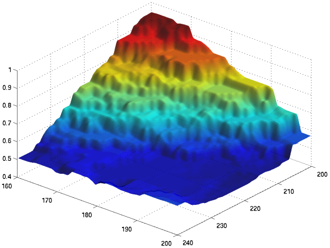

TV regularization

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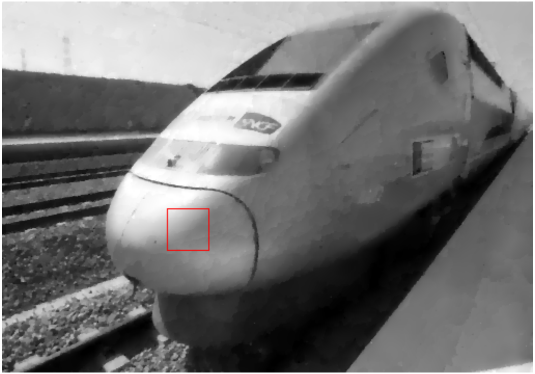

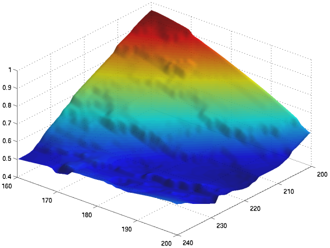

TV-TV^2 infimal-convolution regularization

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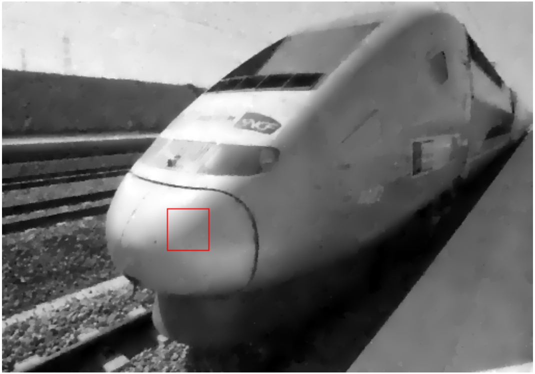

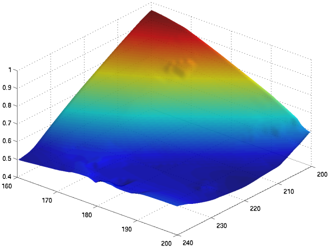

TGV_α^2 regularization

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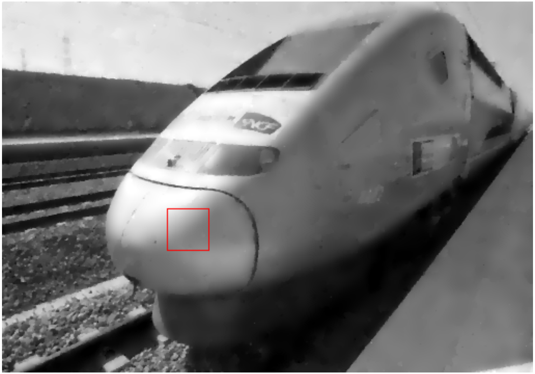

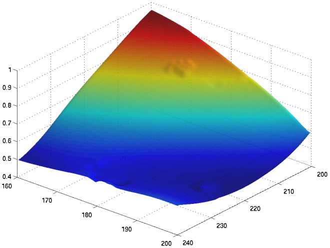

TGV_α^3 regularization

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Are the theoretical results applicable in practice?

- Are there efficient minimization algorithms?
- Does the model lead to improvements in image reconstruction?

Interpretation: TGV of second order

Minimum characterization:

$$\mathrm{TGV}_\alpha^2(u) = \min_{w \in \mathrm{BD}(\Omega)} \alpha_1 \int_\Omega |\nabla u - w| + \alpha_0 \int_\Omega |\mathcal{E}(w)|$$

- $\mathrm{BD}(\Omega) = \{w \in L^1(\Omega, \mathbb{R}^d) \mid \mathcal{E}(w) \in \mathcal{M}(\Omega, \mathrm{Sym}^2(\mathbb{R}^d))\}$
  Vector fields of *bounded deformation*

Intuitive interpretation:

Locally: ∇u smooth
$\rightsquigarrow$ $w = \nabla u \approx$ optimal
$\rightsquigarrow$ $\mathrm{TGV}_\alpha^2 \sim \alpha_0 \int_{\mathrm{loc}} |\nabla^2 u|$

Locally: u jumps
$\rightsquigarrow$ $w = 0 \approx$ optimal
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- Optimal balancing between ∇u and $\nabla^2 u$
  $\rightsquigarrow$ TGV_α^2(pw. smooth) $<$ TGV_α^2(staircases) $\rightsquigarrow$ preferred
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TGV regularization

Joint work with Tuomo Valkonen

Exemplarily:

- Solution of linear ill-posed inverse problems
- Total Generalized Variation of second order

Inverse Problem:

solve $Ku = f$

- $\Omega \subset \mathbb{R}^d$ bounded domain
- $K : L^p(\Omega) \to H$ linear and continuous

Minimize:

Tikhonov-functional

$$\min_{u \in L^p(\Omega)} \frac{\|Ku - f\|_H^2}{2} + \mathrm{TGV}_\alpha^2(u)$$

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- Which conditions ensure existence of solutions?
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- Which conditions ensure existence of solutions?
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$\rightsquigarrow$ Show topological equivalence with $\mathrm{BV}(\Omega)$

Vector fields of bounded deformation

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$$\|w\|_{\mathrm{BD}} = \|w\|_1 + \|\mathcal{E}(w)\|_{\mathcal{M}}$$

- Well-known in the theory of *mathematical plasticity*

Some properties:

- $\mathrm{BD}(\Omega)$ is a Banach space
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  Space of *infinitesimal rigid displacements*
- *Sobolev-Korn inequality*:

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Vector fields of bounded deformation

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Topological equivalence

Theorem: $\Omega \subset \mathbb{R}^d$ sufficiently smooth $\Rightarrow$

$$c\|u\|_{\mathrm{BV}} \leq \|u\|_1 + \mathrm{TGV}_\alpha^2(u) \leq C\|u\|_{\mathrm{BV}} \quad \forall u \in \mathrm{BGV}_\alpha^2(\Omega)$$

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2 Sobolev-Korn inequality + minimum characterization:

$$\begin{aligned}\|\nabla u - Rw\|_{\mathcal{M}} &\leq \|\nabla u - w\|_{\mathcal{M}} + \|w - Rw\|_1 \\ &\leq C_3\big(\alpha_1\|\nabla u - w\|_{\mathcal{M}} + \alpha_0\|\mathcal{E}(w)\|_{\mathcal{M}}\big)\end{aligned}$$

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3 With the help of 1: $\quad \Rightarrow \quad \|u\|_{\mathrm{BV}} \leq C_4\big(\|u\|_1 + \mathrm{TGV}_\alpha^2(u)\big)$

4 Finally: $\mathrm{TGV}_\alpha^2(u) \leq \alpha_1\,\mathrm{TV}(u)$

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Existence and stability

Corollary:

- Coercivity: $\|u - P_1 u\|_{d/(d-1)} \leq C\,\mathrm{TGV}_\alpha^2(u)$

  $P_1 \to \Pi^1$ linear projection on the affine functions Π^1

Theorem:

$$\left.\begin{array}{l} \blacksquare\ 1 < p \leq d/(d-1) \\ \blacksquare\ K : L^p(\Omega) \to H \\ \quad \text{linear and continuous,} \\ \quad H \text{ Hilbert space} \\ \blacksquare\ K \text{ injective on } \Pi^1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \text{Optimization problem} \\ \displaystyle\min_{u \in L^p(\Omega)} \frac{1}{2}\|Ku - f\|^2 \\ \qquad\qquad + \mathrm{TGV}_\alpha^2(u) \\ \text{possesses a solution} \end{array}\right.$$

Proof: Direct method + coercivity of TGV_α^2 $\square$

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Proof: Direct method + coercivity of TGV_α^2 $\square$

Stability: $f^n \to f$ in $H \Rightarrow \begin{cases} u^n \rightharpoonup u \text{ in } L^p(\Omega) \text{ (subseq.)} \\ \mathrm{TGV}_\alpha^2(u^n) \to \mathrm{TGV}_\alpha^2(u) \end{cases}$

General orders

Next step:

- Generalization with respect to k $\rightsquigarrow$ Examine

$$\mathrm{BD}(\Omega, \mathrm{Sym}^k(\mathbb{R}^d)) = \{w \in L^1(\Omega, \mathrm{Sym}^k(\mathbb{R}^d)) \mid \mathcal{E}(w) \in \mathcal{M}(\Omega, \mathrm{Sym}^{k+1}(\mathbb{R}^d))\}$$

$$\|w\|_{\mathrm{BD}} = \|w\|_1 + \|\mathcal{E}(w)\|_{\mathcal{M}}$$

- *Symmetric tensor fields of bounded deformation*

The spaces $\mathrm{BD}(\Omega, \mathrm{Sym}^k(\mathbb{R}^d))$

Theorem: *[B. '11]*

1. $\left.\begin{array}{l} u \in \mathcal{D}(\Omega, \mathrm{Sym}^k(\mathbb{R}^d))^* \\ \text{distribution with } \mathcal{E}(u) = 0 \end{array}\right\} \quad \Rightarrow \quad \nabla^{k+1} \otimes u = 0 \text{ in } \Omega$
2. $\ker(\mathcal{E})$ is a subspace of $\Pi^k(\Omega, \mathrm{Sym}^k(\mathbb{R}^d))$

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Furthermore:

- There is a T such that $\nabla^{k+1} \otimes u = T\mathcal{E}(u)$ for smooth u
- The fundamental solution for $\operatorname{div} \mathcal{E}$ reads as:

$$\Gamma_k^\eta = \sum_{l=0}^{k} (-1)^l \binom{k+1}{l+1} \mathcal{E}^l(\operatorname{div}^l(E_{l+1}\eta))$$

E_m fundamental solution for Δ^m

The spaces $\mathrm{BD}(\Omega, \mathrm{Sym}^k(\mathbb{R}^d))$

Theorem: Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain

1 Trace mapping:

$$\gamma : \mathrm{BD}(\Omega, \mathrm{Sym}^k(\mathbb{R}^d)) \to L^1(\partial\Omega, \mathrm{Sym}^k(\mathbb{R}^d))$$

continuous with respect to strict convergence

2 Gauß-Green theorem:

$$\int_\Omega u \cdot \operatorname{div} v \,\mathrm{d}x = \int_{\partial\Omega} |||(\gamma u \otimes \nu) \cdot v \,\mathrm{d}\mathcal{H}^{d-1} - \int_\Omega v \cdot \,\mathrm{d}\mathcal{E}(u)$$

for $u \in \mathrm{BD}(\Omega, \mathrm{Sym}^k(\mathbb{R}^d))$, $v \in \mathcal{C}^1(\Omega, \mathrm{Sym}^{k+1}(\mathbb{R}^d))$

3 Zero extension: $Eu \in \mathrm{BD}(\mathbb{R}^d, \mathrm{Sym}^k(\mathbb{R}^d))$ with

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1. Embedding result:

$$\mathrm{BD}(\Omega, \mathrm{Sym}^k(\mathbb{R}^d)) \hookrightarrow L^{d/(d-1)}(\Omega, \mathrm{Sym}^k(\mathbb{R}^d))$$

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Consequently:

- Minimum characterization:

$$\mathrm{TGV}_\alpha^k(u) = \min_{\substack{u_l \in \mathrm{BD}(\Omega, \mathrm{Sym}^l(\mathbb{R}^d)), \\ u_0 = u, \ u_k = 0}} \sum_{l=1}^k \alpha_{k-l} \int_\Omega |\mathcal{E}(u_{l-1}) - u_l|$$

- Existence of solutions: $\displaystyle \min_{u \in L^p(\Omega)} \frac{\|Ku - f\|_H^2}{2} + \mathrm{TGV}_\alpha^k(u)$

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$\rightsquigarrow$ "TV **applicable** $\Rightarrow$ TGV **applicable**"

Regularization properties

Joint work with Martin Holler

Theorem: *[B./Holler '13]*

- u^n solution $\sim$ parameters α^n, data f^n,
$$\|f^n - f^\dagger\| \leq \delta_n$$

Multiparameter choice/source condition:

1. $\alpha_i^n \to 0$ and $\delta_n^2/\alpha_i^n \to 0$
2. $\lim_{n\to\infty} \alpha_i^n/\alpha_{i-1}^n > 0$
3. $Ku^\dagger = f^\dagger$ for $u^\dagger \in \mathrm{BV}(\Omega)$

Then: *Convergence:*

- $$\left.\begin{aligned} u^n &\rightharpoonup^* u^* \\ \mathrm{TGV}_{\alpha^*}^{k,l}(u^n) &\to \mathrm{TGV}_{\alpha^*}^{k}(u^*) \end{aligned}\right\} \text{subsequentially}$$
- u^* minimizing-$\mathrm{TGV}_{\alpha^*}^k$ solution

Application: Deconvolution

Problem:

Solve $u * k = f$

- k convolution kernel

Tikhonov functional:

$$\min_{u \in L^2(\Omega)} \frac{1}{2}\|u * k - f\|_2^2 + \mathrm{TGV}_\alpha^2(u)$$

Noisy data f

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TGV_α^2-regularized solution

Optimization methods

Exemplarily: Second-order scalar case $\mathrm{TGV}_\alpha^{2,0} = \mathrm{TGV}_\alpha^2$

$$\min_{u \in L^p(\Omega)} \quad F(u) + \mathrm{TGV}_\alpha^2(u)$$

Approach:

1. Discretize with finite differences
2. Reformulate as convex-concave saddle-point problem
$$\min_{x \in X} \max_{y \in Y} \ \langle Ax, y \rangle + \mathcal{G}(x) - \mathcal{F}^*(y)$$
3. Use primal-dual algorithm of *[Chambolle/Pock '11]*

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$$\begin{cases} y^{n+1} &= (I + \sigma \partial \mathcal{F}^*)^{-1}(y^n + \sigma A \overline{x}^n) \\ x^{n+1} &= (I + \tau \partial \mathcal{G})^{-1}(x^n - \tau A^* y^{n+1}) \\ \overline{x}^{n+1} &= 2x^{n+1} - x^n \end{cases}$$

Saddle-point formulation

- Finite difference approximations ∇^h, $\mathcal{E}^h$, $\mathrm{div}^h = -(\nabla^h)^*$ etc.
- Supremum definition of TGV^2:

$$\min_u \max_v \; F(u) + \langle u, (\mathrm{div}^h)^2 v \rangle - I_{\{\|v\|_\infty \leq \alpha_0\}}(v) - I_{\{\|\omega\|_\infty \leq \alpha_1\}}(\mathrm{div}^h v)$$

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$$x = \begin{bmatrix} u \\ w \end{bmatrix}, \qquad y = \begin{bmatrix} v \\ \omega \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & -\nabla^h \\ -\mathcal{E}^h & -I \end{bmatrix},$$

$$\mathcal{G}(u, w) = F(u), \quad \mathcal{F}^*(v, \omega) = I_{\{\|v\|_\infty \le \alpha_0\}}(v) + I_{\{\|\omega\|_\infty \le \alpha_1\}}(\omega)$$

Primal-dual algorithm 1

Iteration: *[B. '12]*

$$\left.\begin{aligned} \omega^{n+1} &= P_{\{\|\cdot\|_\infty \leq \alpha_1\}}\big(\omega^n + \sigma(\nabla^h \bar{u}^n - \bar{w}^n)\big) \\ v^{n+1} &= P_{\{\|\cdot\|_\infty \leq \alpha_0\}}\big(v^n + \sigma\mathcal{E}^h(\bar{w}^n)\big) \end{aligned}\right\} \text{dual update}$$

$$\left.\begin{aligned} u^{n+1} &= (I + \tau\partial F)^{-1}(u^n + \tau \operatorname{div}^h \omega^{n+1}) \\ w^{n+1} &= w^n + \tau(\operatorname{div}^h v^{n+1} + \omega^{n+1}) \end{aligned}\right\} \text{primal update}$$

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- $P_{\{\|\cdot\|_\infty \le \alpha_0\}}$, $P_{\{\|\cdot\|_\infty \le \alpha_1\}}$ amount to pointwise operations
- $(I + \tau\partial F)^{-1}$ resolvent mapping $\rightsquigarrow$ assumed to be known
- Converges for appropriate choice of $\sigma, \tau > 0$

[Chambolle/Pock '11]

Applications

Examples for Algorithm 1:

$F(u)$	$(I+\tau\partial F)^{-1}(u)$
$\frac{1}{2}\Vert u-f\Vert_H^2$	$\frac{u+\tau f}{1+\tau}$
$\Vert u-f\Vert_1$	$f+\mathbf{S}_\tau(u-f)$ $\mathbf{S}_\tau$ soft-shrinkage operator
$\frac{1}{2}\Vert Ku-f\Vert_H^2$	$(I+\tau K^*K)^{-1}(u+\tau K^*f)$ solve, e.g., with CGNE

- Primary application: Denoising problems

Applications

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$F(u)$	$(I + \tau\partial F)^{-1}(u)$
$\frac{1}{2}\Vert u - f\Vert_H^2$	$\frac{u + \tau f}{1 + \tau}$
$\Vert u - f\Vert_1$	$f + \mathbf{S}_\tau(u - f)$ $\mathbf{S}_\tau$ soft-shrinkage operator
$\frac{1}{2}\Vert Ku - f\Vert_H^2$	$(I + \tau K^*K)^{-1}(u + \tau K^* f)$ solve, e.g., with CGNE

- Primary application: Denoising problems

Applications

Examples for Algorithm 1:

$F(u)$	$(I + \tau\partial F)^{-1}(u)$
$\frac{1}{2}\Vert u - f\Vert_H^2$	$\frac{u + \tau f}{1 + \tau}$
$\Vert u - f\Vert_1$	$f + \mathbf{S}_\tau(u - f)$ $\mathbf{S}_\tau$ soft-shrinkage operator
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Primal-dual algorithm 2

Alternative:

- Additional dual variable: $F(u) = \max_p \langle Ku, p\rangle + \tilde{F}(u) - G(p)$
- Needs only resolvents w.r.t. ∂G, $\partial \tilde{F}$, not $(I + \tau \partial F)^{-1}$

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Iteration: *[B. '12]*

$$\begin{cases} p^{n+1} = (I + \sigma\partial G)^{-1}(p^n + \sigma K\bar{u}^n) \\ \omega^{n+1} = P_{\{\|\cdot\|_\infty \leq \alpha_1\}}\big(\omega^n + \sigma(\nabla^h \bar{u}^n - \bar{w}^n)\big) \\ v^{n+1} = P_{\{\|\cdot\|_\infty \leq \alpha_0\}}\big(v^n + \sigma\mathcal{E}^h(\bar{w}^n)\big) \end{cases}$$

$$\begin{cases} u^{n+1} = (I + \tau\partial\tilde{F})^{-1}\big(u^n + \tau(\operatorname{div}^h \omega^{n+1} - K^* p^{n+1})\big) \\ w^{n+1} = w^n + \tau(\operatorname{div}^h v^{n+1} + \omega^{n+1}) \end{cases}$$

$$\left\{ \bar{u}^{n+1} = 2u^{n+1} - u^n, \quad \bar{w}^{n+1} = 2w^{n+1} - w^n \right.$$

Applications

Examples for Algorithm 2:

$F(u)$	$G(p)$	$(I + \sigma\partial G)^{-1}(p)$
$\frac{1}{2}\Vert Ku - f\Vert_H^2$	$\frac{\Vert p\Vert^2}{2} + \langle f, p\rangle$	$\frac{p - \sigma f}{1 + \sigma}$
$\Vert Ku - f\Vert_1$	$I_{\{\Vert p\Vert_\infty \leq 1\}}(p) + \langle f, p\rangle$	$P_{\{\Vert p\Vert_\infty \leq 1\}}(p - \sigma f)$

- Here: $\tilde{F}(u) = 0 \Rightarrow (I + \tau\partial\tilde{F})^{-1}(u) = u$
- Primary application: TGV^2-regularized solution of $Ku = f$

Applications

Examples for Algorithm 2:

$F(u)$	$G(p)$	$(I + \sigma\partial G)^{-1}(p)$
$\frac{1}{2}\|Ku - f\|_H^2$	$\frac{\|p\|^2}{2} + \langle f, p\rangle$	$\frac{p - \sigma f}{1 + \sigma}$
$\|Ku - f\|_1$	$I_{\{\|p\|_\infty \leq 1\}}(p) + \langle f, p\rangle$	$P_{\{\|p\|_\infty \leq 1\}}(p - \sigma f)$

- Here: $\tilde{F}(u) = 0 \Rightarrow (I + \tau\partial\tilde{F})^{-1}(u) = u$
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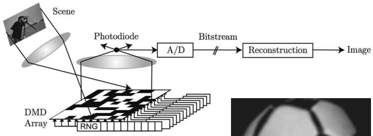
Outline

Application: Compressive imaging



"Single-pixel camera"

Original image

Problem:

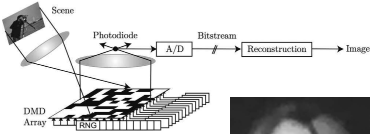
- Reconstruct incomplete data with respect to a given basis

Variational formulation:

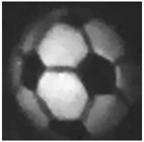
$$\min_{Au=f} \; R(u)$$

- A basis analysis operator
- R "sparsifying" penalty

Application: Compressive imaging



"Single-pixel camera"



1600 measurements

Problem:

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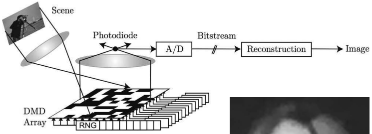
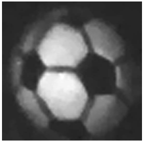
Compressed sensing:

- $R(u) = \|u\|_1$
- Captures solution with minimal "L^0-norm" with high probability

Application: Compressive imaging



"Single-pixel camera"



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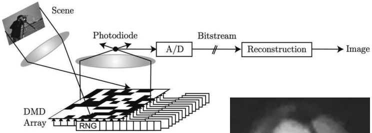
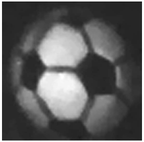
In particular:

- $R(u) = \mathrm{TV}(u)$ "Gradient sparsity"

Application: Compressive imaging



"Single-pixel camera"



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In particular:

- $R(u) = \mathrm{TV}(u)$
  "Gradient sparsity"
  $\rightsquigarrow$ **Use** TGV **as penalty**

Example

Test data:

- From "Rice Single-Pixel Camera Project"

  http://dsp.rice.edu/cscamera

- Reconstruction from varying number of samples

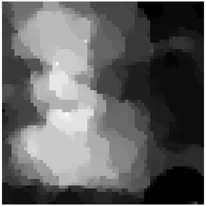
Algorithm:

- Primal-dual method 2

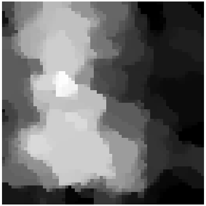
TV-based reconstruction:

768 samples

384 samples



256 samples



192 samples

Example

TGV-based reconstruction:

Test data:

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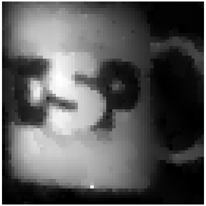
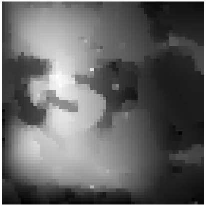
  http://dsp.rice.edu/cscamera
- Reconstruction from varying number of samples

Algorithm:

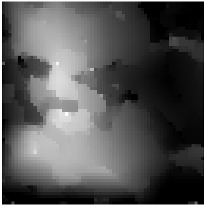
- Primal-dual method 2



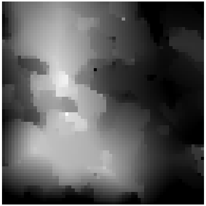
768 samples



384 samples



256 samples



192 samples

Application: Image decompression

Joint work with Martin Holler

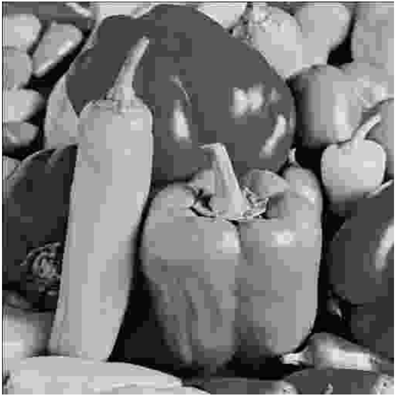
Original image (8 bpp)

JPEG compression scheme:

- Lossy procedure
- High compression
  $\rightsquigarrow$ disturbing artifacts (ringing, blocking)

Application: Image decompression

Joint work with Martin Holler



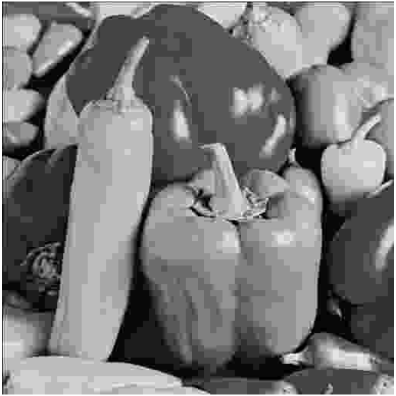
JPEG image (0.062 bpp)

JPEG compression scheme:

- Lossy procedure
- High compression
  $\rightsquigarrow$ disturbing artifacts (ringing, blocking)

Application: Image decompression

Joint work with Martin Holler



JPEG image (0.062 bpp)

JPEG compression scheme:

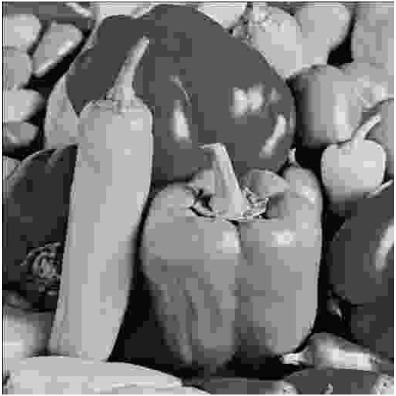
- Lossy procedure
- High compression
  - $\rightsquigarrow$ disturbing artifacts (ringing, blocking)

Goal:

- Remove artifacts
- Respect given information

Application: Image decompression

Joint work with Martin Holler



JPEG image (0.062 bpp)

JPEG compression scheme:

- Lossy procedure
- High compression
  $\rightsquigarrow$ disturbing artifacts (ringing, blocking)

Goal:

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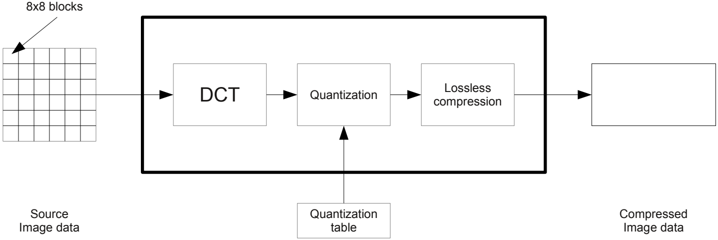
- Respect given information

$\rightsquigarrow$ **Use** TGV **image model**

JPEG decompression model

JPEG compression scheme:

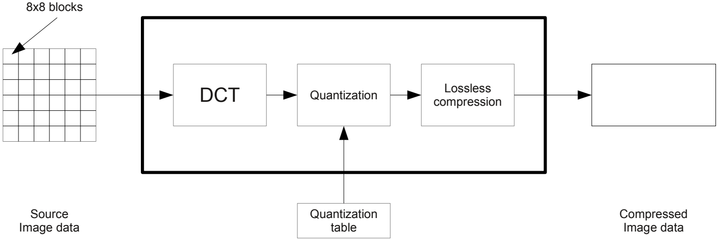


Problem:

- Many images give same JPEG object
  $\rightsquigarrow$ convex set C
- Standard decompression
  $\rightsquigarrow$ particular choice $\rightsquigarrow$ artifacts

JPEG decompression model

JPEG compression scheme:



Problem:

- Many images give same JPEG object
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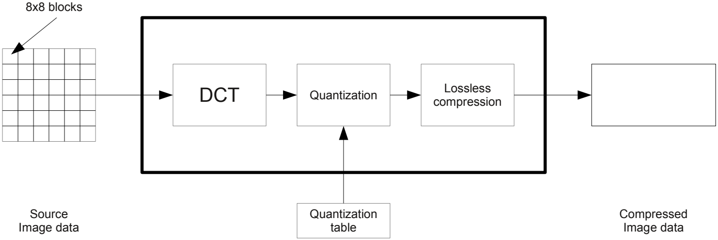
- Standard decompression
  $\rightsquigarrow$ particular choice $\rightsquigarrow$ artifacts

Idea: Optimize over all possible choices

JPEG decompression model

JPEG compression scheme:



Problem:

- Many images give same JPEG object $\rightsquigarrow$ convex set C
- Standard decompression $\rightsquigarrow$ particular choice $\rightsquigarrow$ artifacts

Idea: Optimize over all possible choices

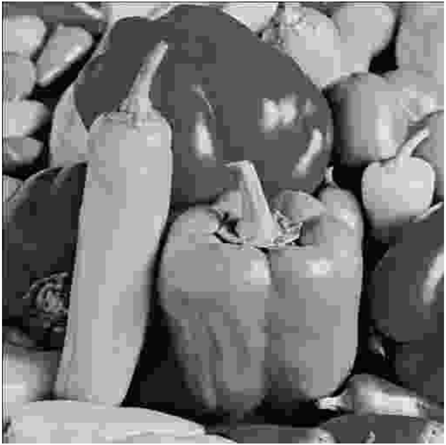
TGV **Model:**

$$\min_{u \in L^2(\Omega)} \mathrm{TGV}_\alpha^2(u) + I_C(u)$$

JPEG decompression: Example

Decompression of grayscale images: 0.062 bpp

standard decompression



JPEG decompression: Example

Decompression of grayscale images: 0.062 bpp

JPEG-TGV decompression

JPEG decompression: Example

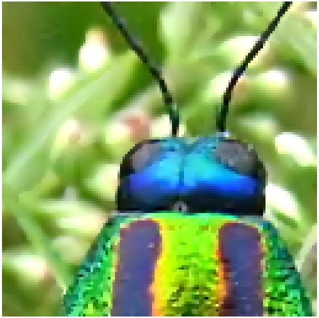
Decompression of color images: 0.051 bpp

standard decompression

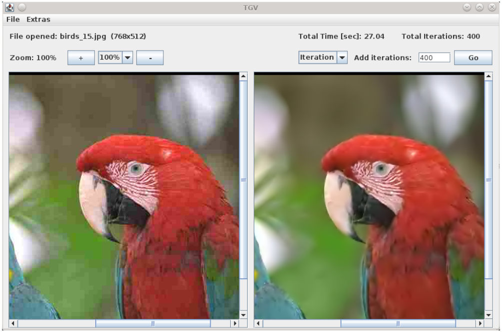
JPEG decompression: Example

Decompression of color images: 0.051 bpp

JPEG-TGV decompression



Towards real-life application



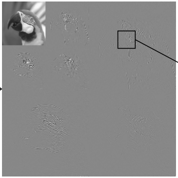
Software:

- Handles all flavors of JPEG (grayscale/color, chroma subsampling, etc.)
- Fast OpenMP + GPU (CUDA) implementation
- Interactive applet available

Extension to JPEG 2000

JPEG 2000 compression scheme:

W



00001010 =10
00000100 =4
00000111 =7
00000001 =1
00000100 =4
00000010 =2
00010000 =16
01011100 =92
00000011 =3

10000000011

Uncompressed image

Wavelet-Transformed image

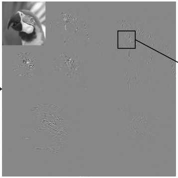
Bit-level coding

JPEG2000 file

Extension to JPEG 2000

JPEG 2000 compression scheme:

W →



→

00001010 =10
00000100 =4
00000111 =7
00000001 =1
00000100 =4
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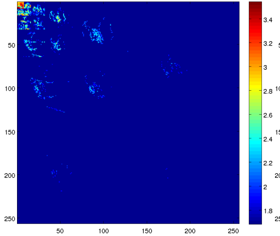
01011100 =92
00000011 =3

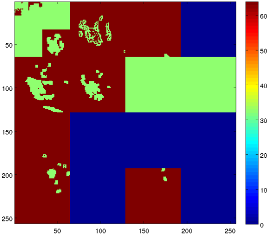
→ 10000000011

Uncompressed image | Wavelet-Transformed image | Bit-level coding | JPEG2000 file

Source image set:

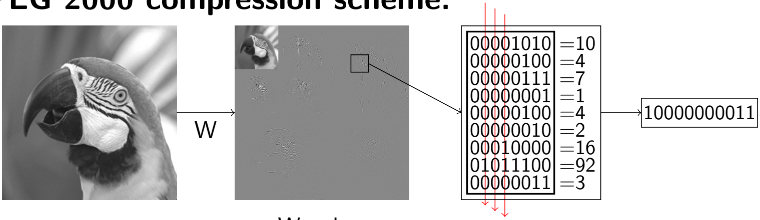
- Same structure





Extension to JPEG 2000

JPEG 2000 compression scheme:

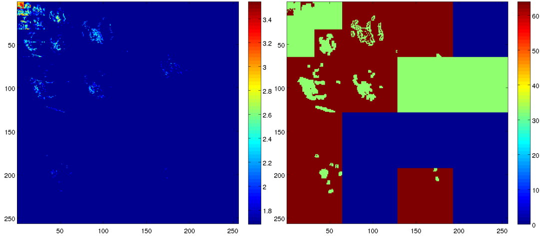


Uncompressed image | Wavelet-Transformed image | Bit-level coding | JPEG2000 file

Source image set:

- Same structure

⇝ TGV-based decompression can also be applied



JPEG 2000: Example

Decompression of color images:

0.019 bpp

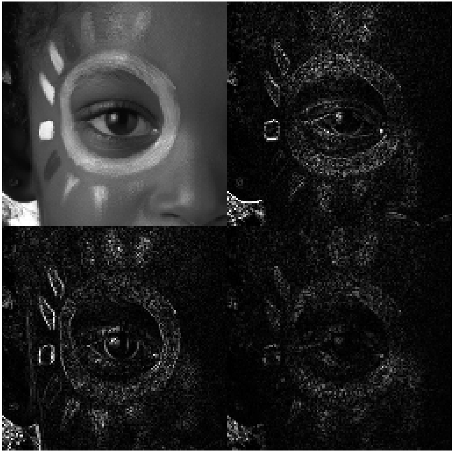
standard decompression

JPEG 2000: Example

Decompression of color images: 0.019 bpp

JPEG2000-TGV decompression

Free extra: Wavelet zooming



JPEG 2000:

- Approximation coefficients (+ precision)
- Some wavelet coefficients (+ precision)

Wavelet zooming:

- Approximation coefficients (full precision)
- No wavelet coefficients

$\rightsquigarrow$ same framework can be used

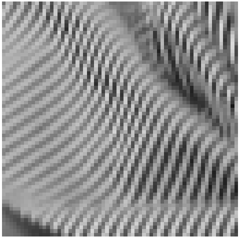
Wavelet zooming: Example

Test data:

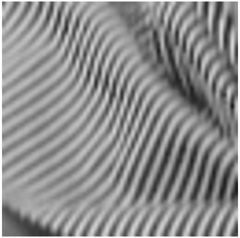
- Barbara's headscarf

Zooming:

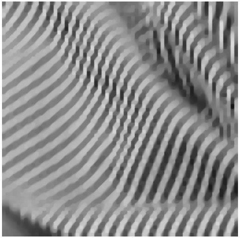
- $64 \times 64 \to 256 \times 256$



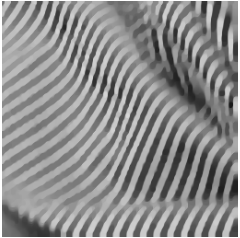
constant interp.



cubic interp.



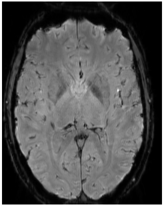
Haar wavelet+TGV

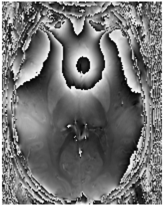


CDF wavelet+TGV

Quantitative susceptibility mapping

Joint work with Christian Langkammer





$\downarrow$

χ

Motivation:

- Measure magnetic susceptibility χ with MRI
  $\rightsquigarrow$ quantification of specific biomarkers
- Can be obtained from 3D GRE phase data
  $\rightsquigarrow$ reconstruction is a challenging problem

State of the art:

- Multi-step reconstruction procedure
- Last step: Regularized solution of a deconvolution problem

Aims:

- Regularize with TGV
- Develop efficient algorithm

Standard approach

Three-step procedure:

1. Unwrap phase data $\varphi_0^{\text{wrap}} \to \varphi_0$

2. Subtract harmonic background field, e.g.

$$\min_{\varphi^{\text{bg}}} \ \tfrac{1}{2}\|\varphi^{\text{bg}} - \varphi_0\|_2^2 \quad \text{subject to} \quad \Delta\varphi^{\text{bg}} = 0$$

$\varphi^{\text{qsm}} = \varphi_0 - \varphi^{\text{bg}}$ for optimal φ^{bg}

3. Perform regularized deconvolution

$$\min_{\chi} \ \tfrac{1}{2}\|\chi * \delta - c\varphi^{\text{qsm}}\|_2^2 + \alpha R(\chi)$$

$$(\mathcal{F}\delta)(k_x, k_y, k_z) = \frac{\frac{1}{3}(k_x^2 + k_y^2) - \frac{2}{3}k_z^2}{k_x^2 + k_y^2 + k_z^2}, \qquad c = \frac{1}{2\pi T_E \gamma B_0}$$

optimal $\chi \rightsquigarrow$ susceptibility map

Standard approach

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optimal $\chi \rightsquigarrow$ susceptibility map

$\rightsquigarrow$ **Is a single-step variational approach possible?**

Integrative variational modelling

Ingredients:

- Applying Δ to the inverse problem $\chi * \delta = c\varphi^{\mathrm{qsm}}$ yields wave-equation-like partial differential equation:

$$\Box\chi = \frac{1}{3}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2\frac{\partial^2}{\partial z^2}\Big)\chi = c\Delta\varphi^{\mathrm{qsm}}$$

- The background field is harmonic:

$$\Rightarrow \quad \Delta\varphi^{\mathrm{qsm}} = \Delta\varphi_0 \quad \text{on brain mask } \Omega'$$

- $\Delta\varphi_0$ can be obtained from the wrapped phase:

$$\Delta\varphi_0 = \mathrm{Imag}\big((\Delta e^{i\varphi_0^{\mathrm{wrap}}})e^{-i\varphi_0^{\mathrm{wrap}}}\big)$$ *[Schofield/Zhu '03]*

Integrative variational modelling

Ingredients:

- Applying Δ to the inverse problem $\chi * \delta = c\varphi^{\mathrm{qsm}}$ yields wave-equation-like partial differential equation:

$$\square\chi = \frac{1}{3}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2\frac{\partial^2}{\partial z^2}\Big)\chi = c\Delta\varphi^{\mathrm{qsm}}$$

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Solve: $\Box\chi = c\Delta\varphi_0 \quad \text{in} \quad \Omega'$

The variational problem

Objective functional:

- Discrepancy: $\frac{1}{2}\|\psi\|_2^2$

  with $\quad \Delta\psi = \Box\chi - c\Delta\varphi_0 \quad$ on brain mask Ω'
- Regularization of χ: TGV of second order

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Integrative TGV-QSM reconstruction:

$$\begin{cases} \displaystyle\min_{\chi,\psi} \ \frac{1}{2}\int_{\Omega'} |\psi|^2 \,\mathrm{d}x + \mathrm{TGV}_\alpha^2(\chi) \\ \text{subject to } \Delta\psi = \Box\chi - c\Delta\varphi_0 \text{ on } \Omega' \end{cases}$$

- Numerical method: Primal-dual algorithm 2
- Essentially a one-step approach

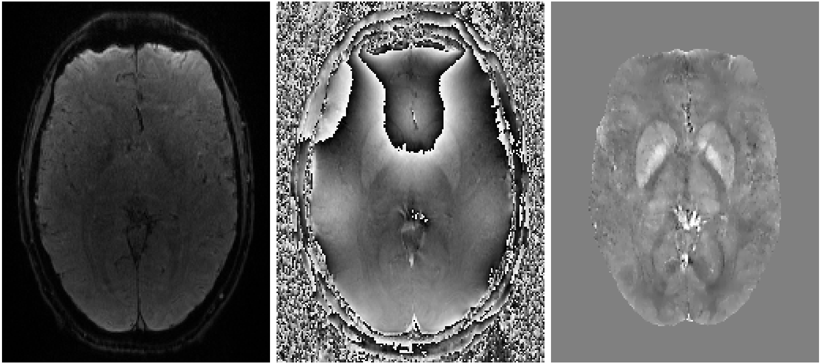
  $\rightsquigarrow$ robust with respect to noise

Numerical example

TGV-QSM reconstruction:

- 3D EPI, resolution 1mm^3, size 230x230x176, **TA 29 sec**

magnitude | phase | TGV-QSM



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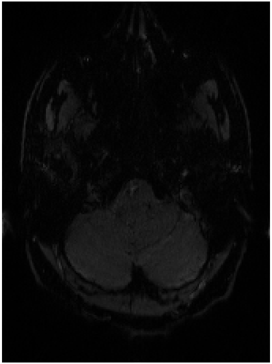
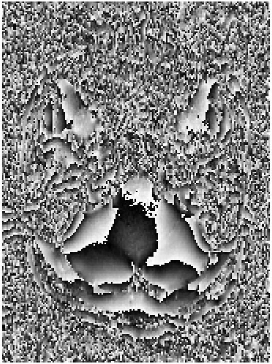
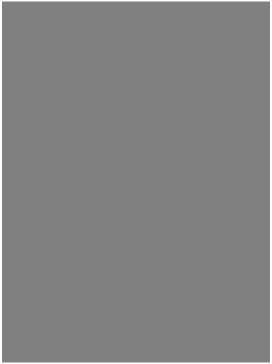
magnitude



phase

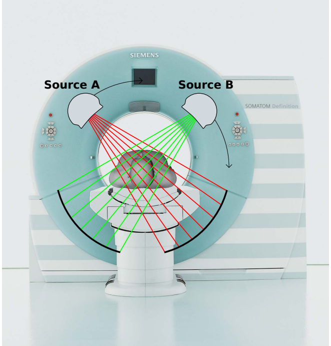


TGV-QSM



Application: Dual energy CT

Joint work with Michael Pienn

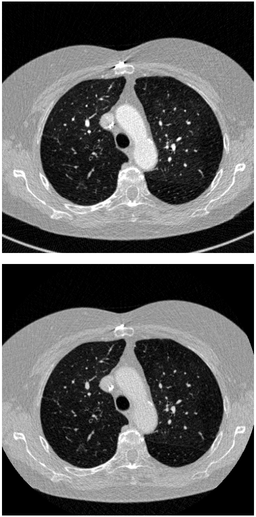


Dual energy CT:

- Two X-ray sources with different spectra
- Two images are acquired
- Allows to differentiate and quantify contrast agent concentration

- Facilitates diagnosis in many cases
- Reconstructions are noisy due to low dose

Assessment of lung perfusion



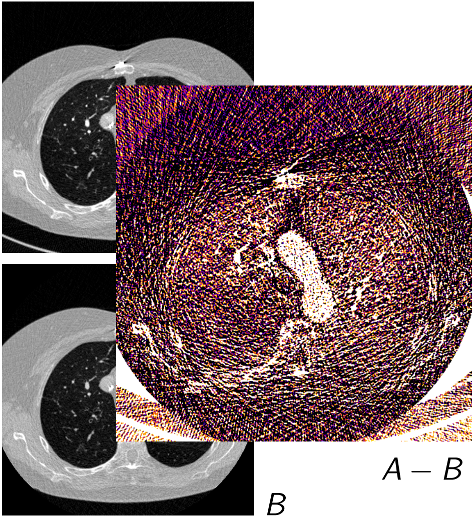
A

B

Important application:

- Diagnosis of *pulmonary embolism*
- Contrast agent concentration lower in affected areas of the lung

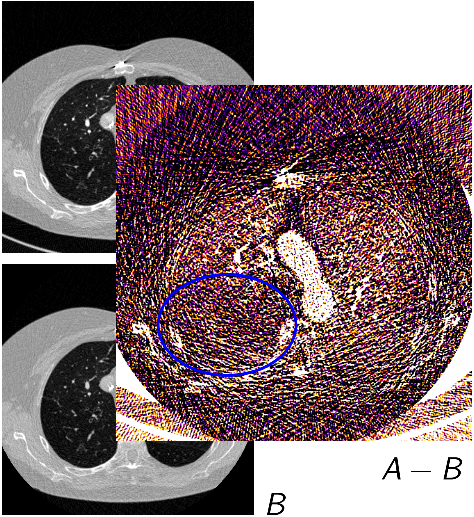
Assessment of lung perfusion



Important application:

- Diagnosis of *pulmonary embolism*
- Contrast agent concentration lower in affected areas of the lung
- Can be seen in the difference image

Assessment of lung perfusion



B

$A - B$

Important application:

- Diagnosis of *pulmonary embolism*
- Contrast agent concentration lower in affected areas of the lung
- Can be seen in the difference image

Dual energy CT: Denoising

Problem setup:

- Given: Two noisy image sequences A_0, B_0 (3D data set)
- Base + difference image $\rightsquigarrow$ BGV^2-images
- Prevent contrast change $\rightsquigarrow$ Use L^1 discrepancy

Minimization problem:

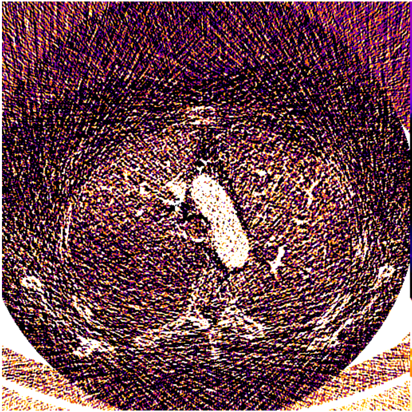
$$\min_{(A,B)\in L^1(\Omega)^2} \|A - A_0\|_1 + \|B - B_0\|_1 + \mathrm{TGV}^2_\alpha(B) + \mathrm{TGV}^2_\alpha(A - B)$$

Numerical realization:

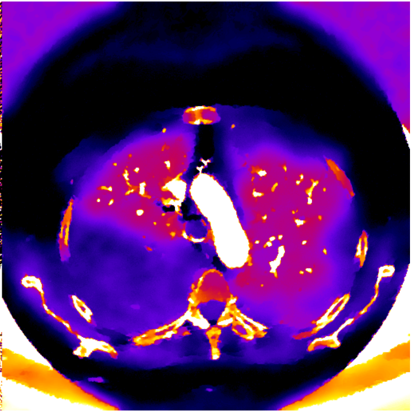
- Primal-dual algorithm 1 (with a slight modification)

Dual energy CT: Example

$A_0 - B_0$

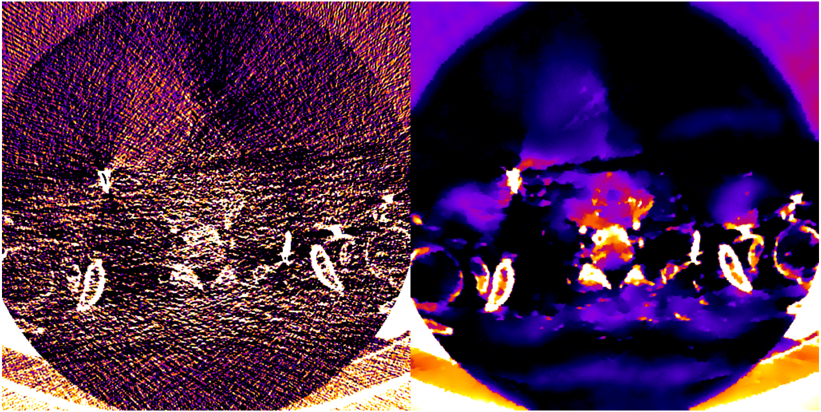


$A - B$



Dual energy CT: Example

$A_0 - B_0$ $\qquad$ $A - B$



Outline

Summary

- Total generalized variation
  - Consistent model for piecewise smooth images
  - Functional-analytic framework for regularization of inverse problems is available
- Computational methods
  - Two variants of a flexible primal-dual algorithm
  - Easy to implement & suitable for parallelization

Imaging applications:

1. Denoising and deblurring
2. Compressive imaging
3. JPEG(2000) decompression and wavelet zooming

Medical applications:

1. Quantitative susceptibility mapping
2. Dual energy CT

$\rightsquigarrow$ high-quality reconstructions

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