Continuous Basis Pursuit

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Ubiquitous signal property: Superposition of prototype features, at arbitrary (**continuous**) times/positions/orientations/sizes etc ...

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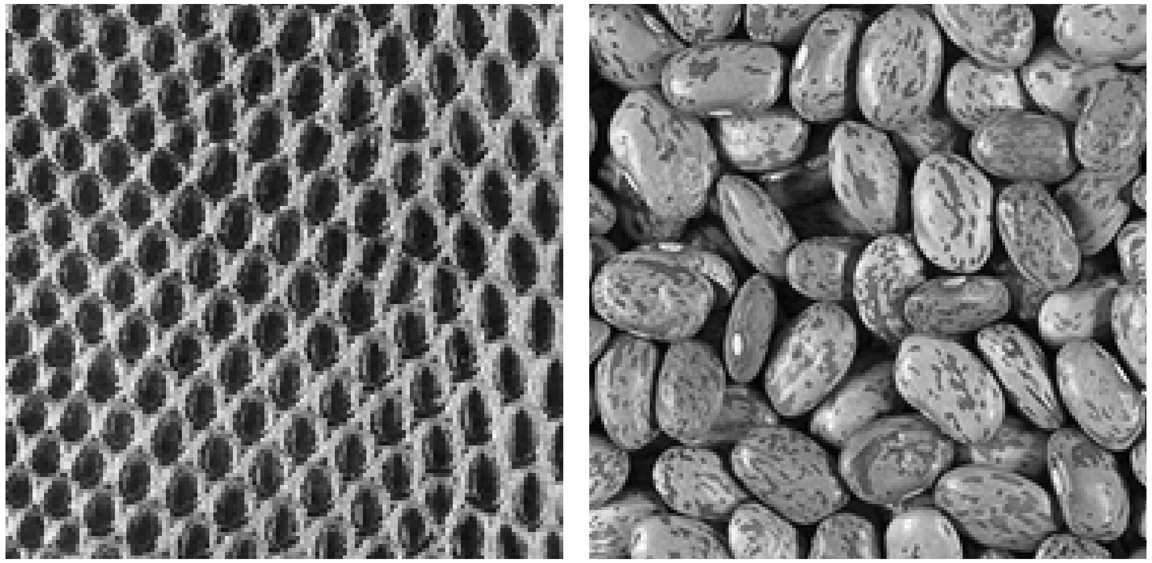
images (e.g., texture)



Ubiquitous signal property: Superposition of prototype features, at arbitrary (**continuous**) times/positions/orientations/sizes etc ...

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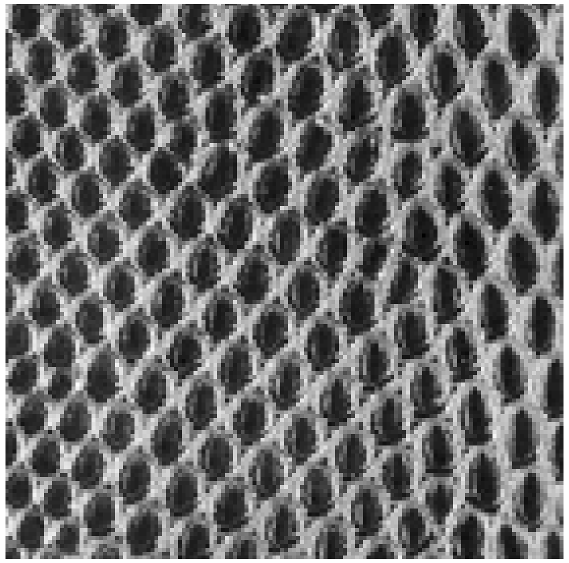

sounds (e.g., speech / music)

radar or seismic signals

spikes in neurons

Continuous sparse inverse problem:

$$y(t) = \sum_j a_j f(t - \tau_j) + \eta(t)$$

f localized waveform, assume known (for now)

τ_j independent arrival times

a_j i.i.d., distributed according to $p(a)$

$\eta(t)$ AWGN, variance σ^2 (for now)

Goal: recover a_j's and τ_j's

Classical solutions (from the 1940’s) for related problems...

- Matched filter
  (isolated occurrences, known waveform)

- Wiener-Kolmogorov filter
  (dense occurrences, known waveform)

Modern sparse inverse solution

- Re-formulate as a **discrete** linear inverse problem with dictionary of Δ-shifted functions:

$$\{f(t - n\Delta)\}$$

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- Enforce sparsity with a convex objective function for coefficients $\vec{x}$:

$$\arg\min_{\vec{x}} ||y(t) - \sum_{n=1}^{N} x_n f(t-n\Delta)||_2^2 + \lambda||\vec{x}||_1$$

LASSO [Tibshirani, 1996]   Basis Pursuit [Chen, Donoho, Sanders, 1998]

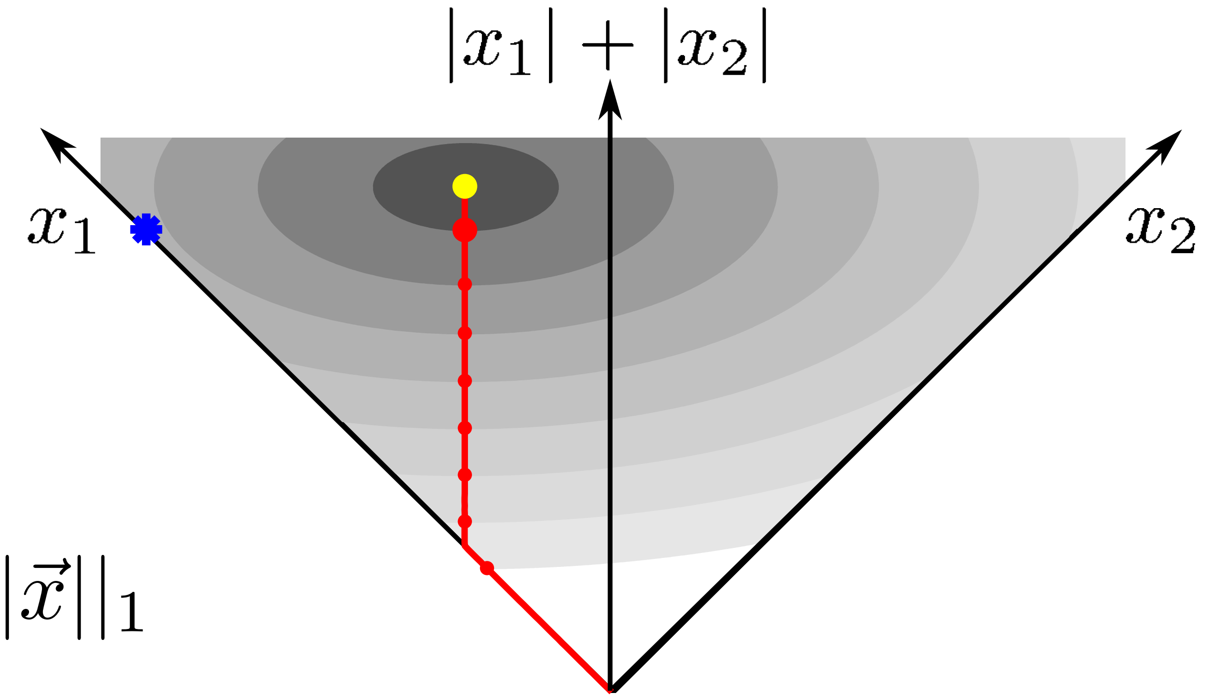
Drawbacks of BP

$$\arg\min_{\vec{x}} ||y(t) - \sum_{n=1}^{N} x_n f(t - n\Delta)||_2^2 + \lambda ||\vec{x}||_1$$

- Δ should be small
  - approximation of f at non-integer $\{\tau_j\}$
  - Mapping of $\{x_n\}$ to $\{a_j, \tau_j\}$?
- But Δ can't be too small
  - Basis redundancy => convex approximation fails to produce sparsity [Candes, Romberg, Tau, 2006]
  - High computational cost

2D illustration of objective function and solutions

$$||y(t) - \sum_{n=1}^{N} x_n f(t - n\Delta)||_2^2 + \lambda||\vec{x}||_1$$



gray - reconstruction error

blue - L0 solution

yellow - LS solution ($\lambda = 0$)

red - solutions for different λ

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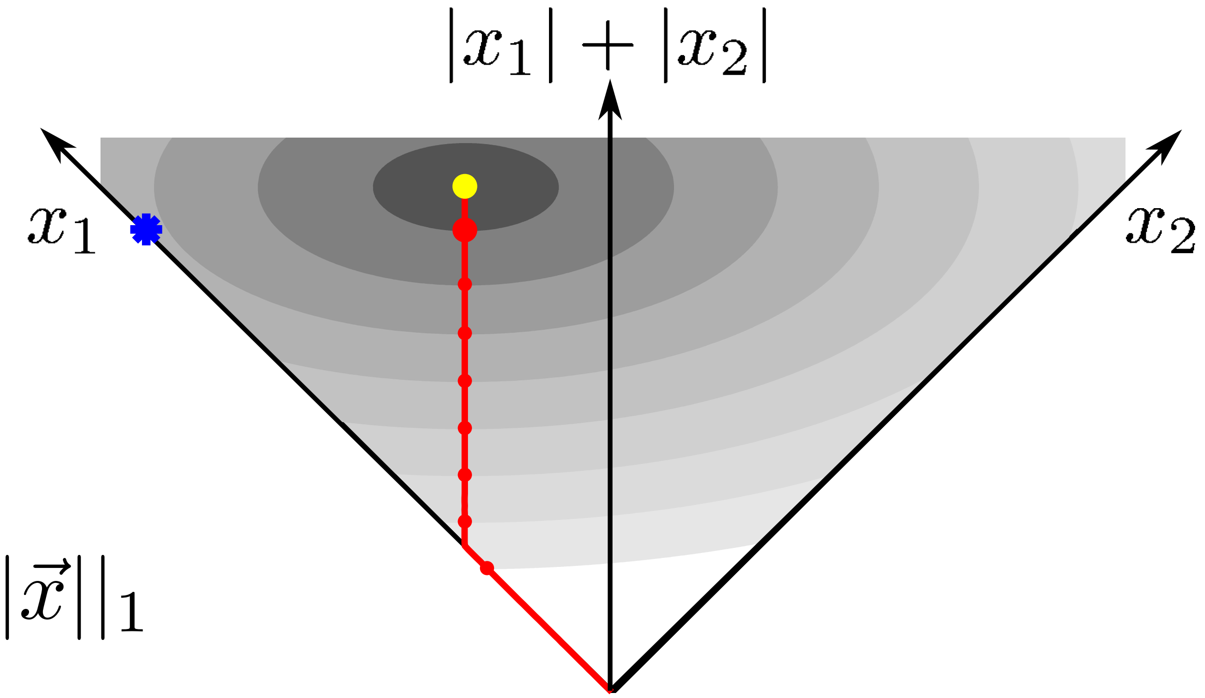

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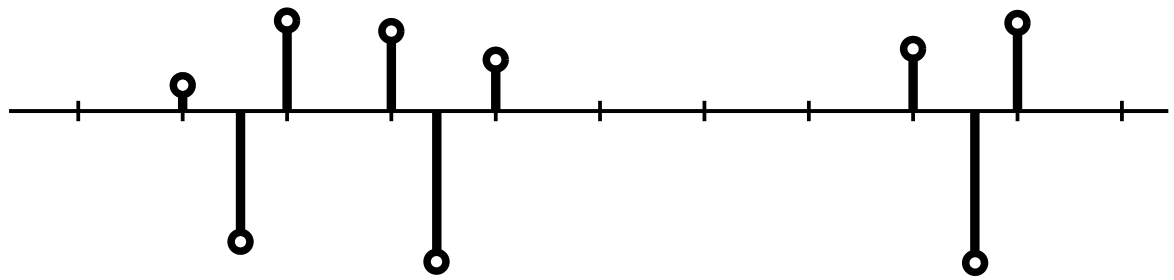
Signal simulation



upward: BP solution
downward: true waveform locations/amplitudes

Continuous basis pursuit - Taylor

Choose spacing such that:

$$af(t+\tau) \approx af(t) + a\tau f'(t), \qquad |\tau| < \Delta/2$$

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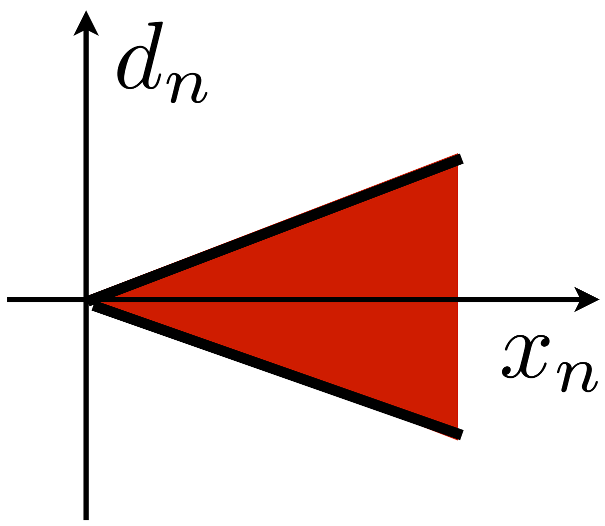

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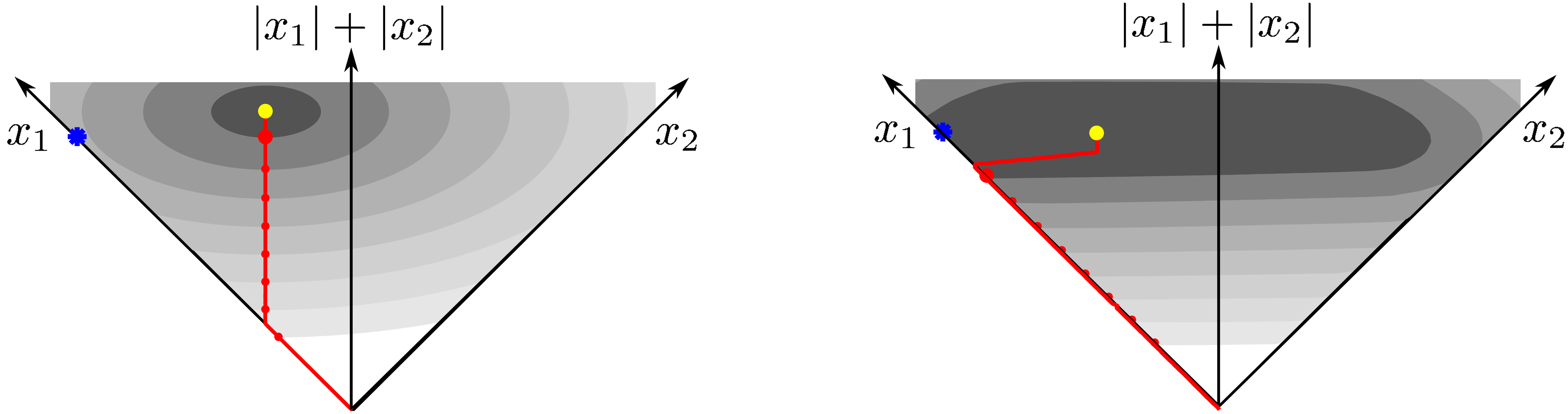
with $|d_n| < |x_n|\Delta/2$

BP | CBP - Taylor1



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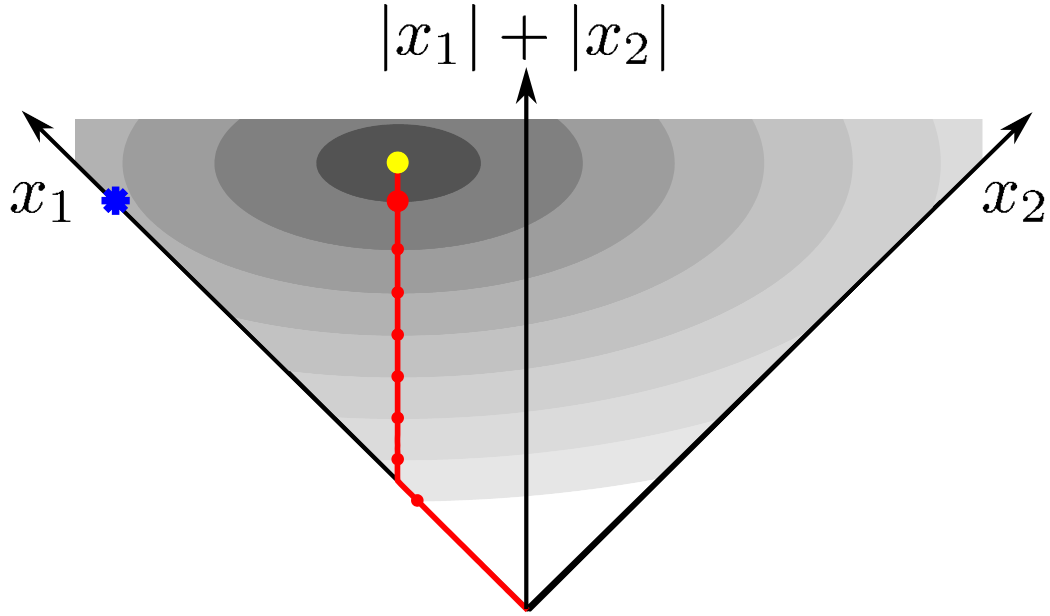

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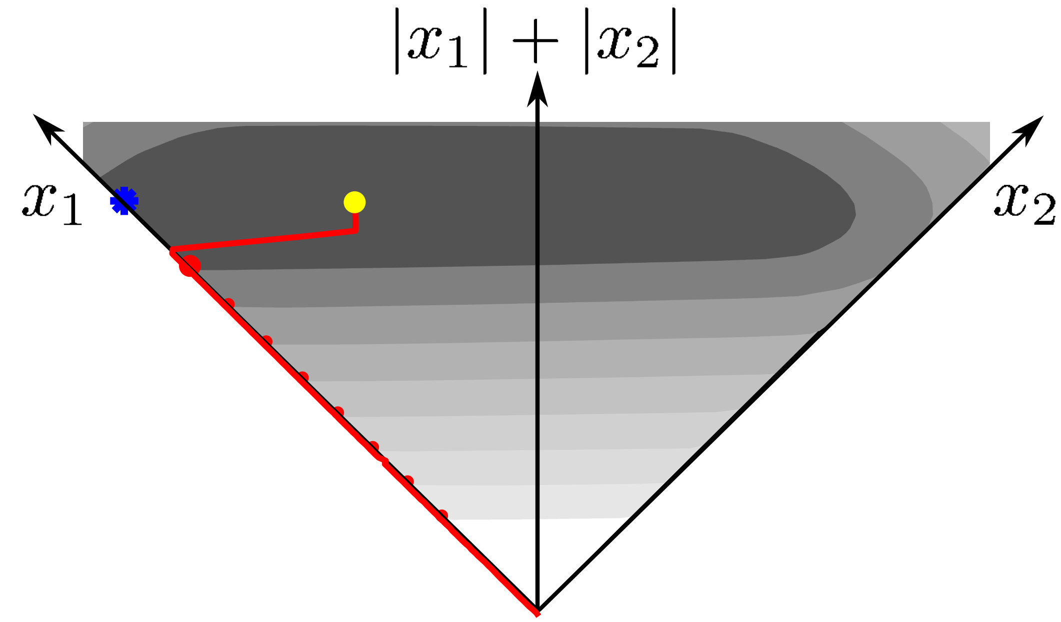

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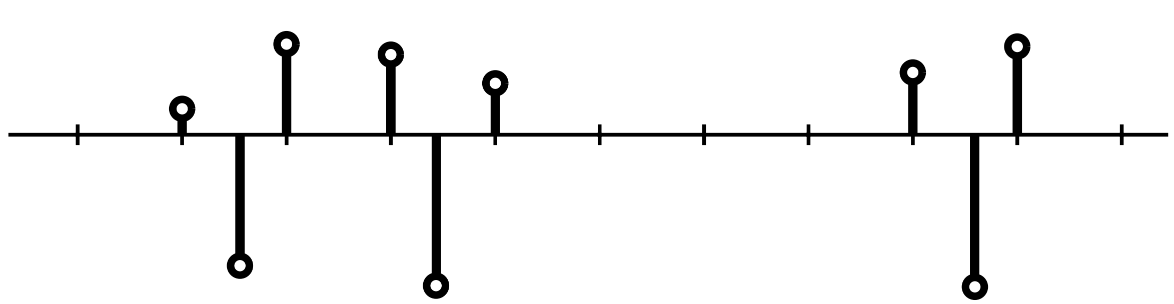

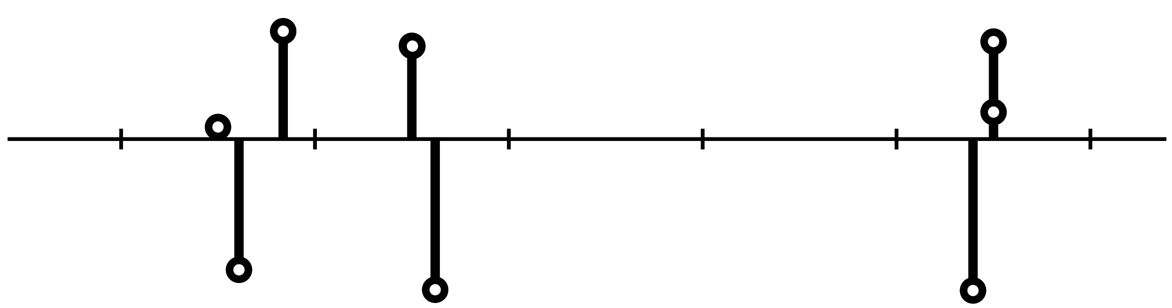

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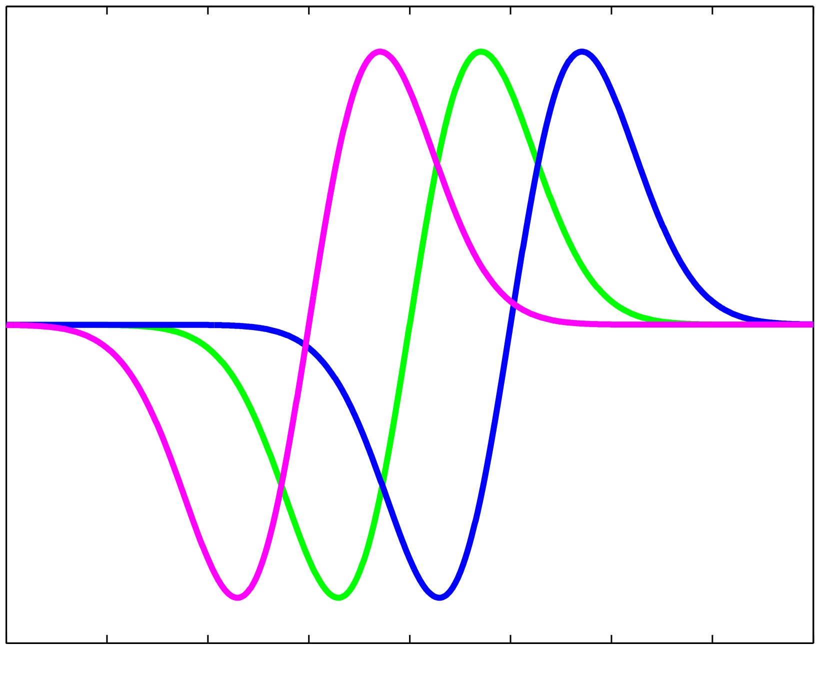

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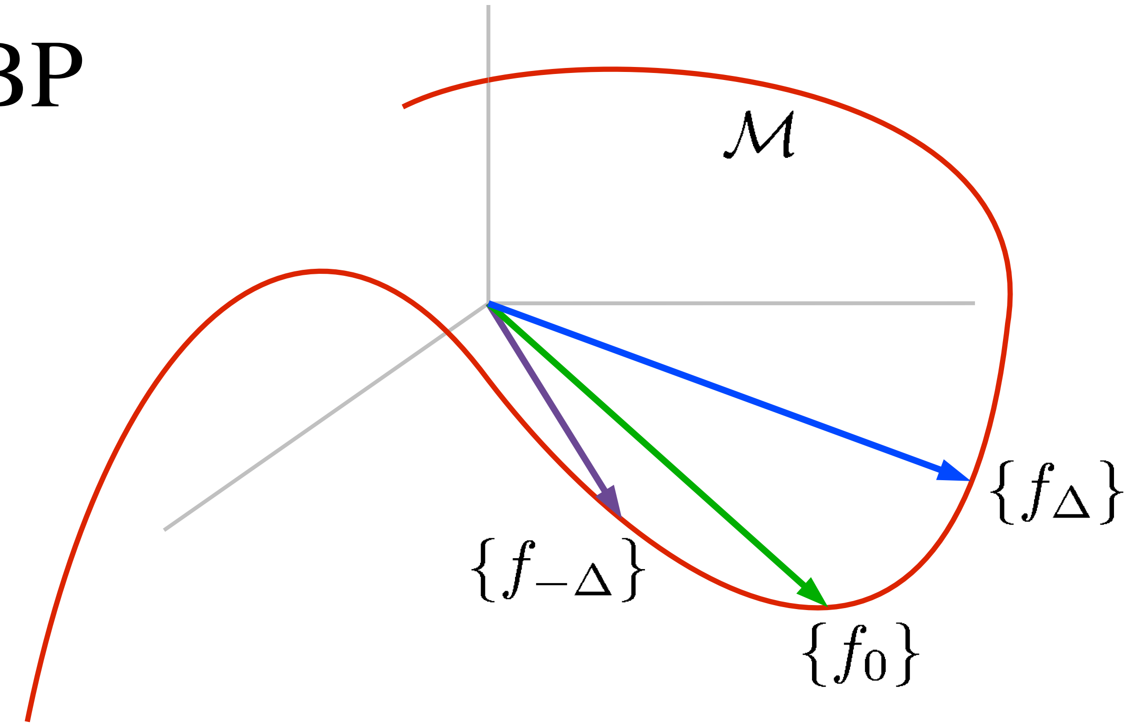

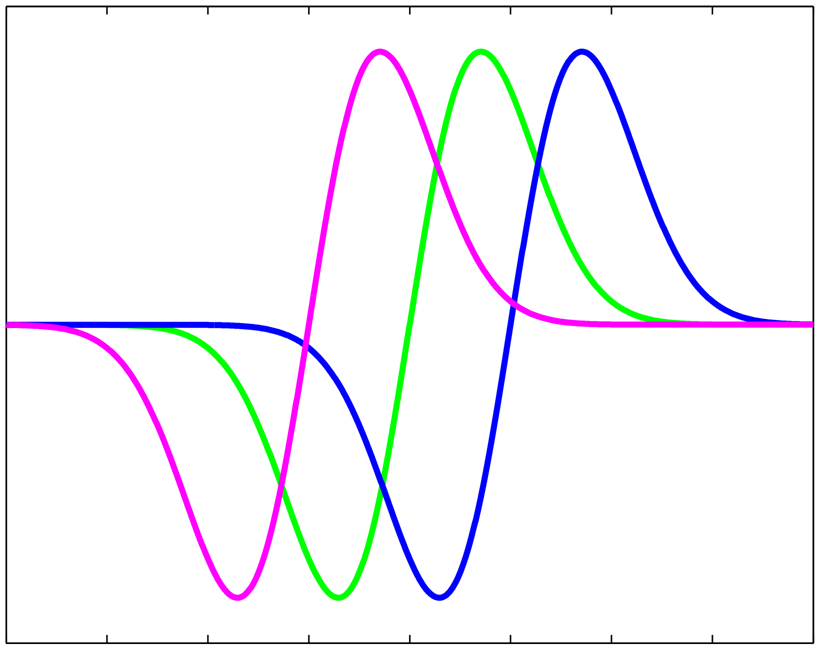

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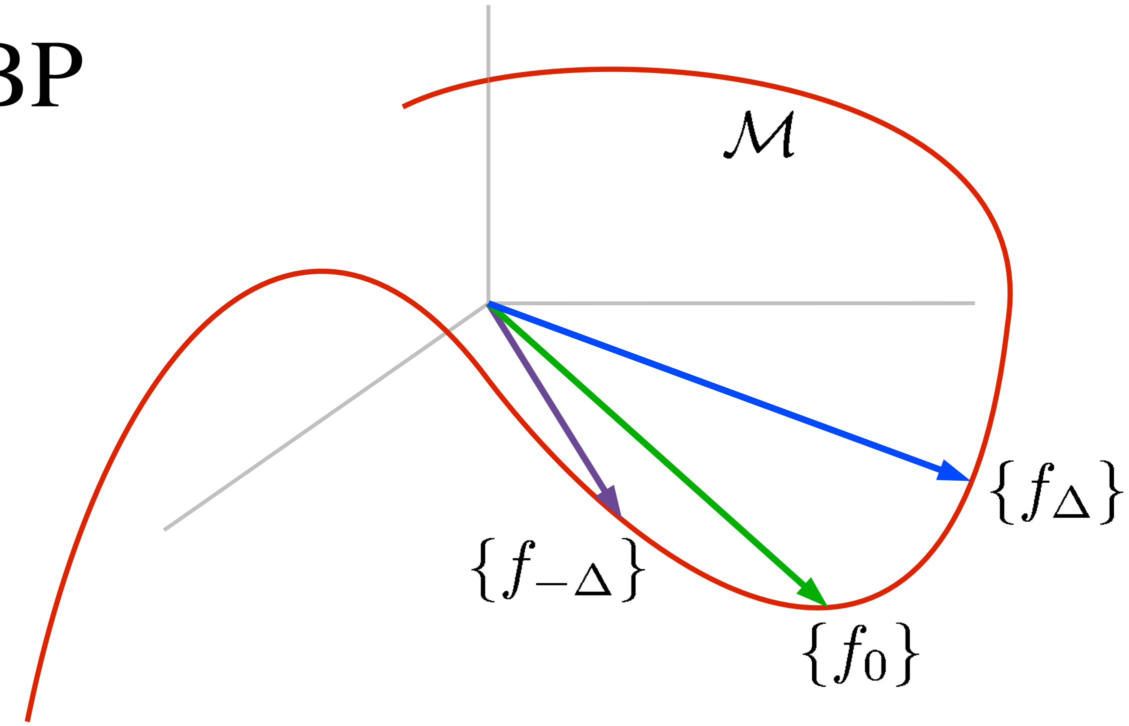

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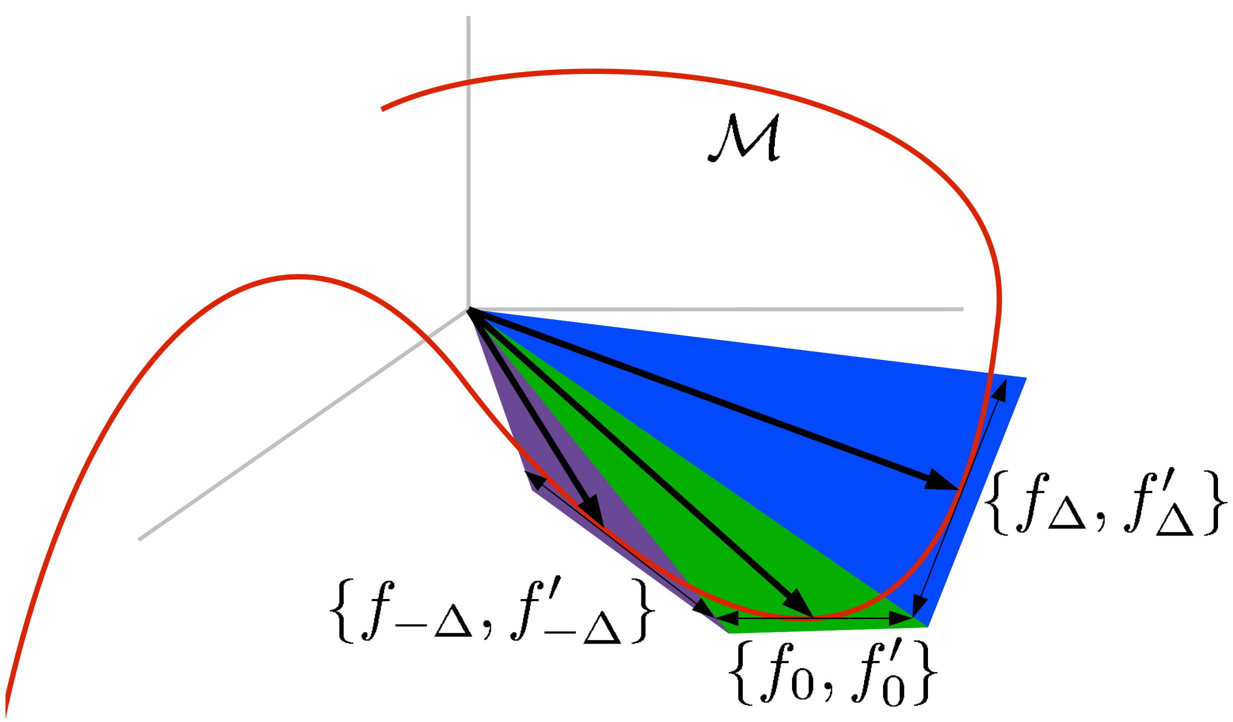
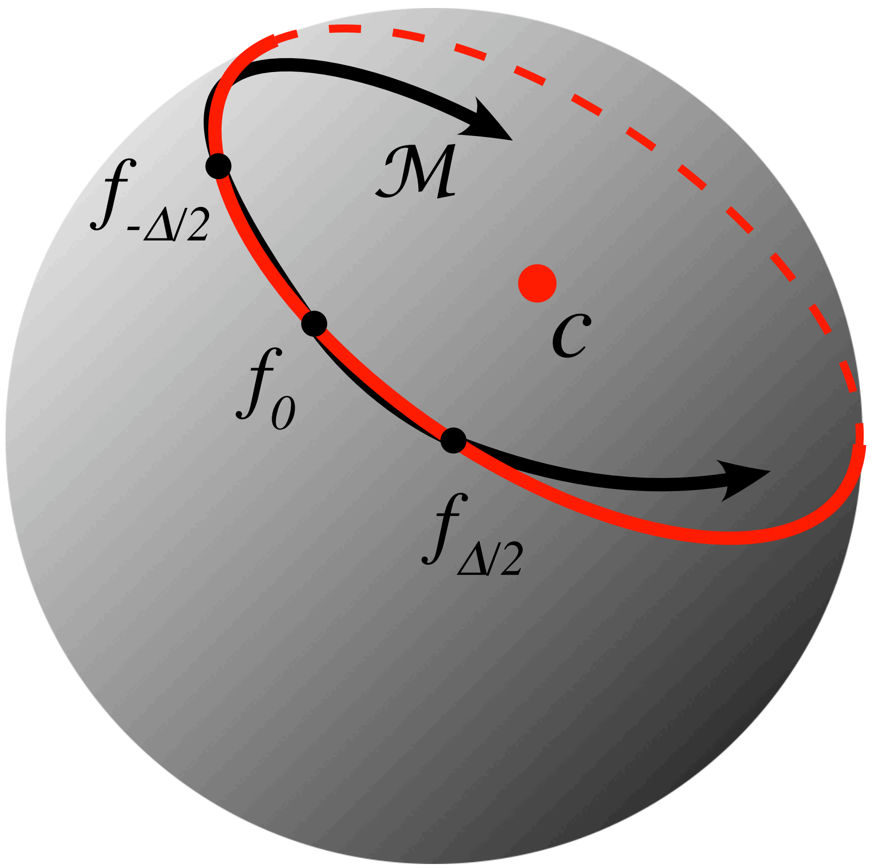


Polar interpolation



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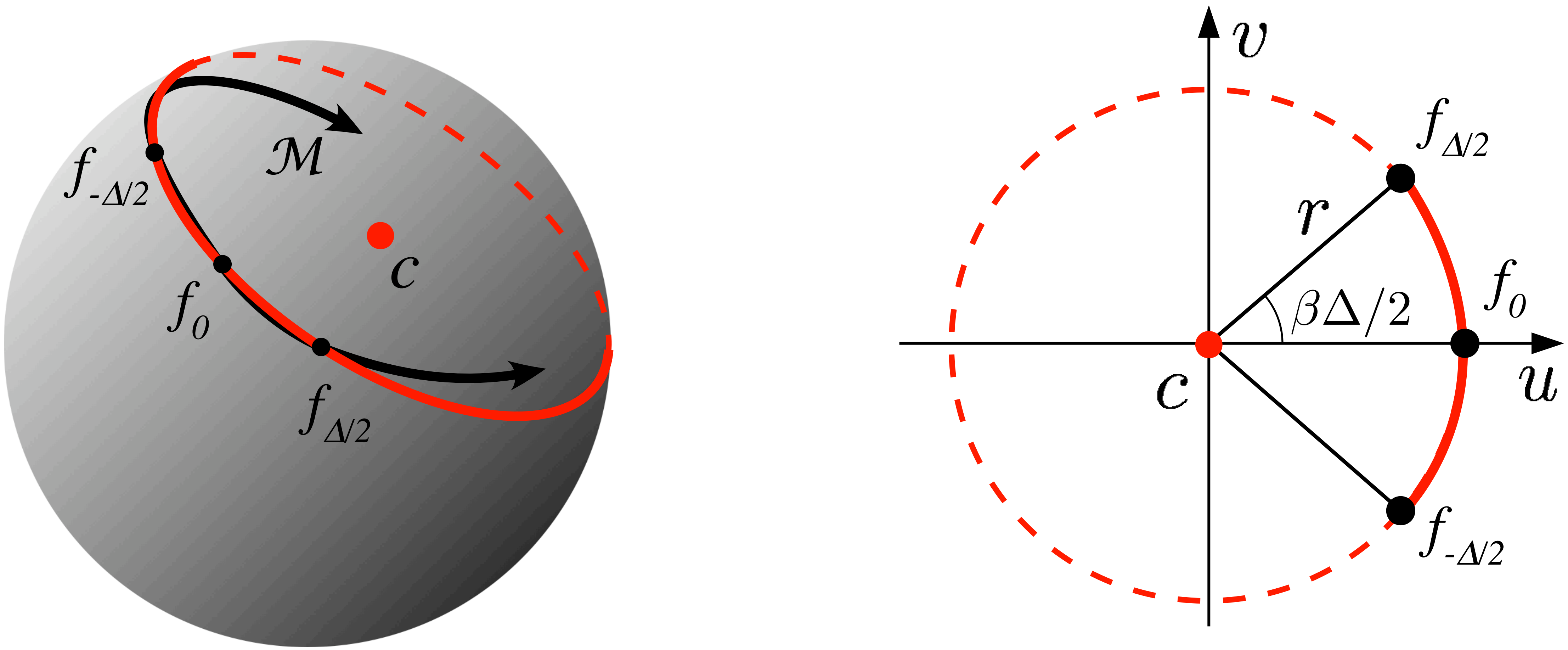

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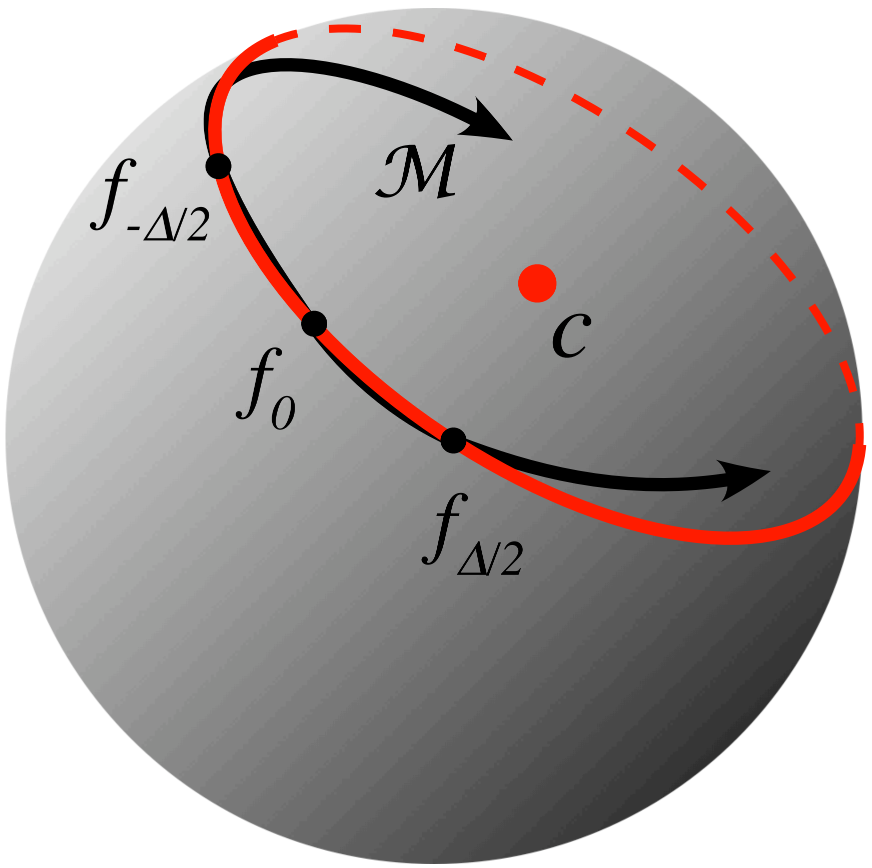
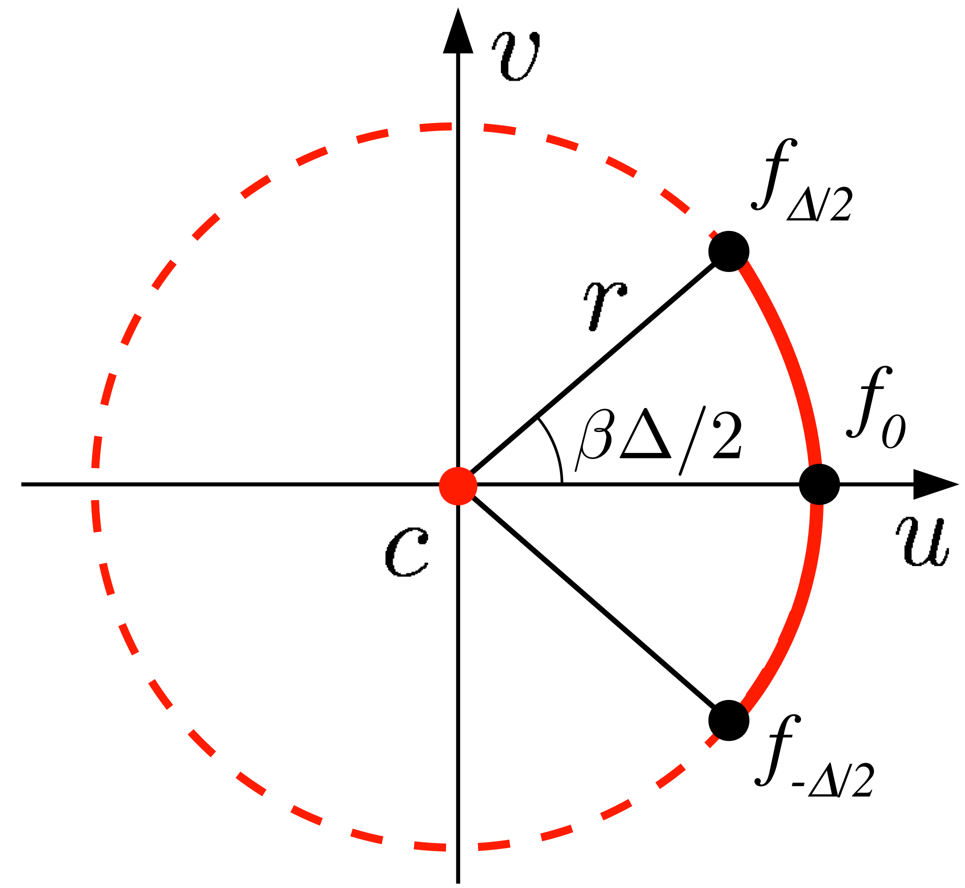
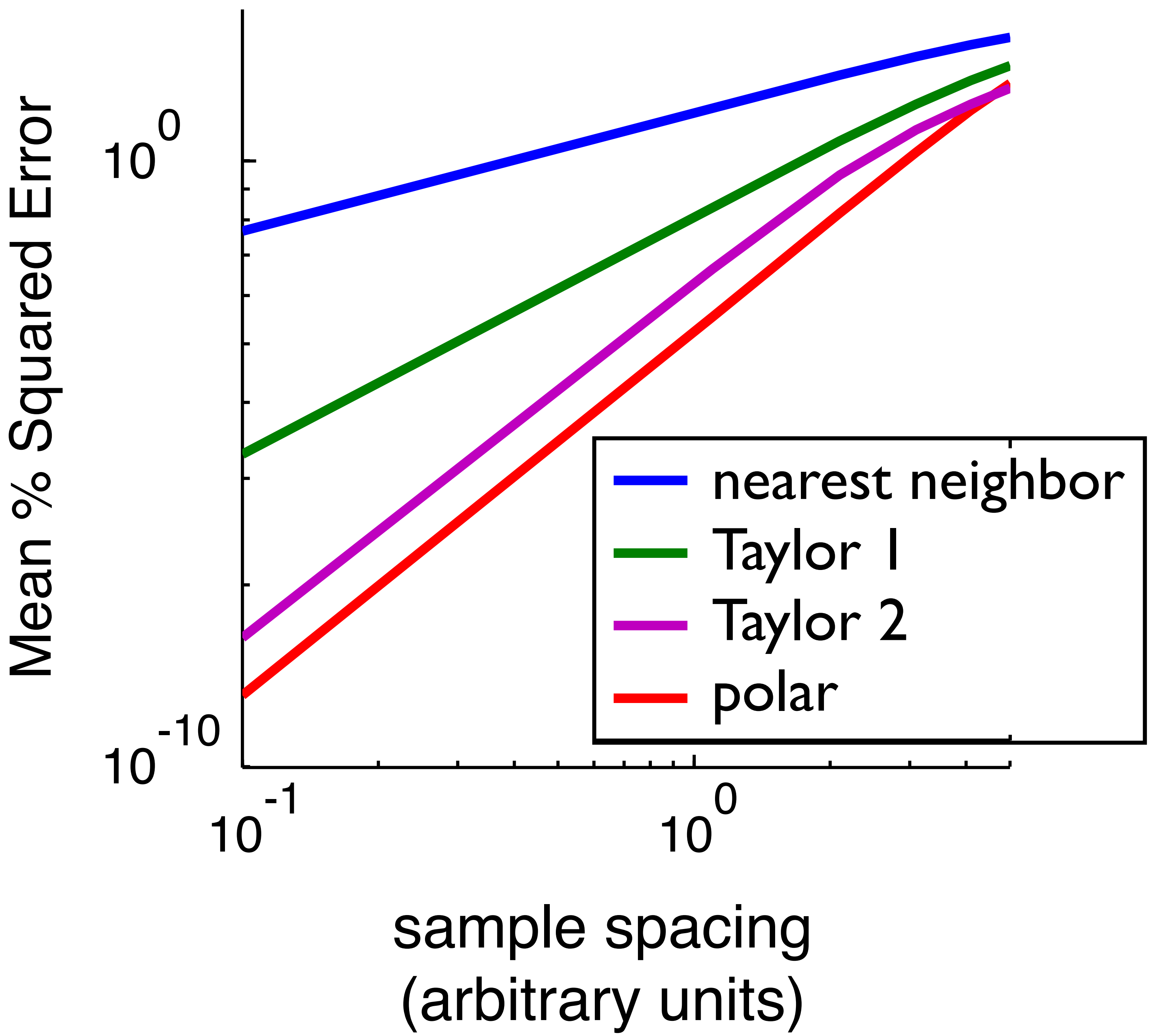




$$f(t+\tau) \approx c(t) + r\cos(\beta\tau)u(t) + r\sin(\beta\tau)v(t), \quad |\tau| < \Delta/2$$

$$\begin{pmatrix} f(t-\frac{\Delta}{2}) \\ f(t) \\ f(t-\frac{\Delta}{2}) \end{pmatrix} = \begin{pmatrix} 1 & r\cos(\beta\Delta/2) & -r\sin(\beta\Delta/2) \\ 1 & r & 0 \\ 1 & r\cos(\beta\Delta/2) & r\sin(\beta\Delta/2) \end{pmatrix} \begin{pmatrix} c(t) \\ u(t) \\ v(t) \end{pmatrix}$$

Interpolator convergence rates



Continuous basis pursuit - polar

Choose $\{\Delta, \beta, r\}$ such that

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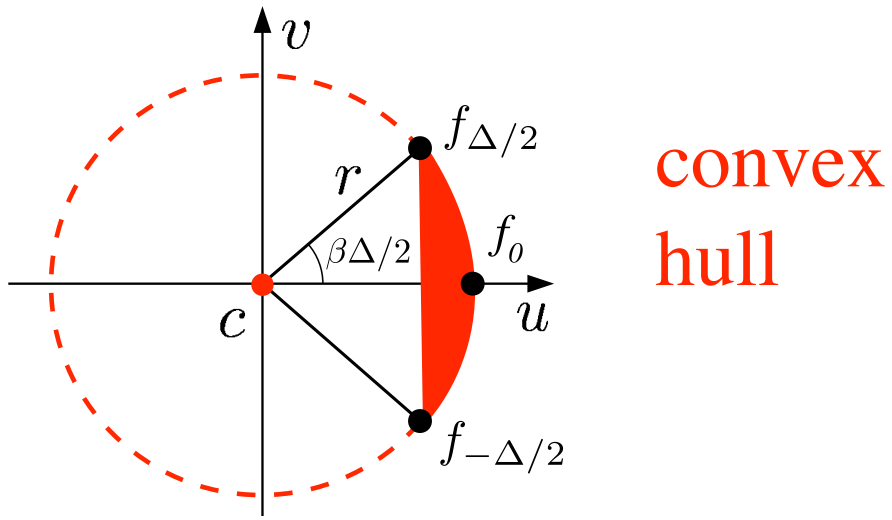
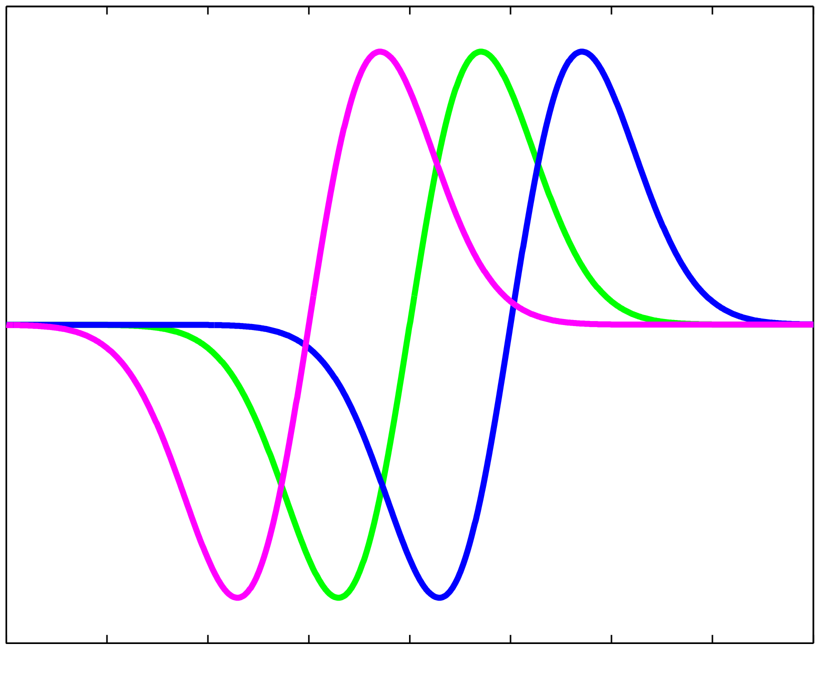
Solve:

$$\arg\min_{\vec{x},\vec{w},\vec{z}} ||y(t) - \sum_{n=1}^{N} x_n c(t-n\Delta) + w_n u(t-n\Delta) + z_n v(t-n\Delta)||_2^2 + \lambda||\vec{x}||_1$$

with
$$x_n \geq 0,$$
$$\sqrt{w_n^2 + z_n^2} \leq r x_n,$$
$$x_n rcos(\beta\Delta/2) \leq w_n$$





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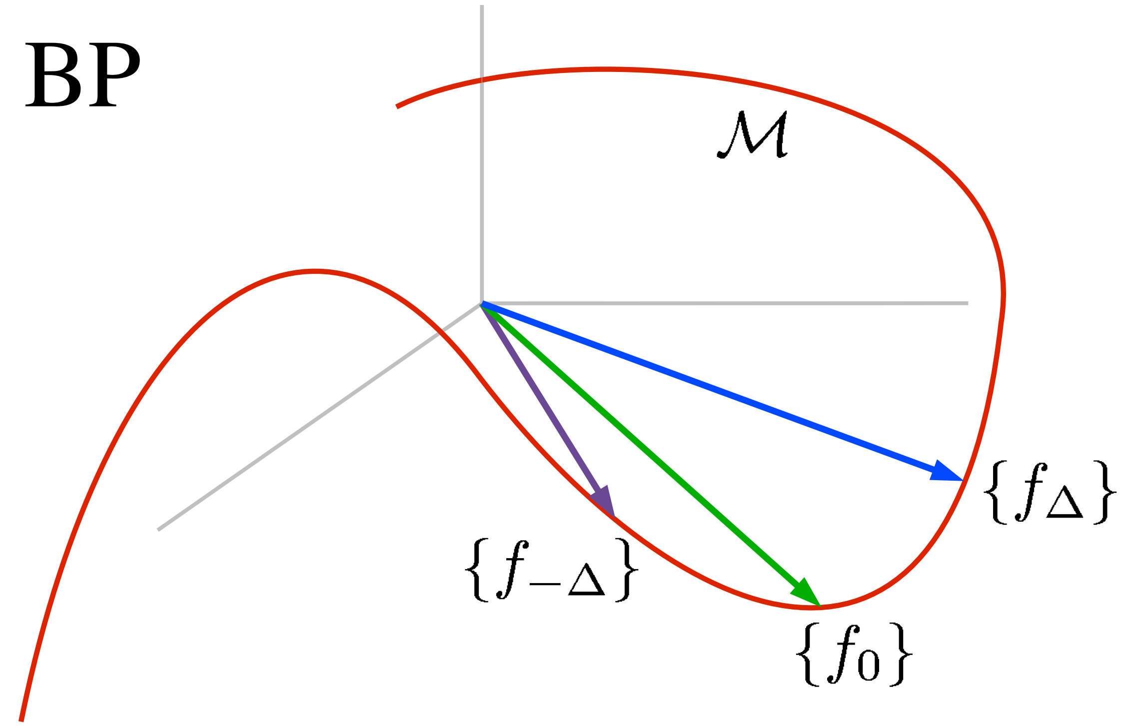
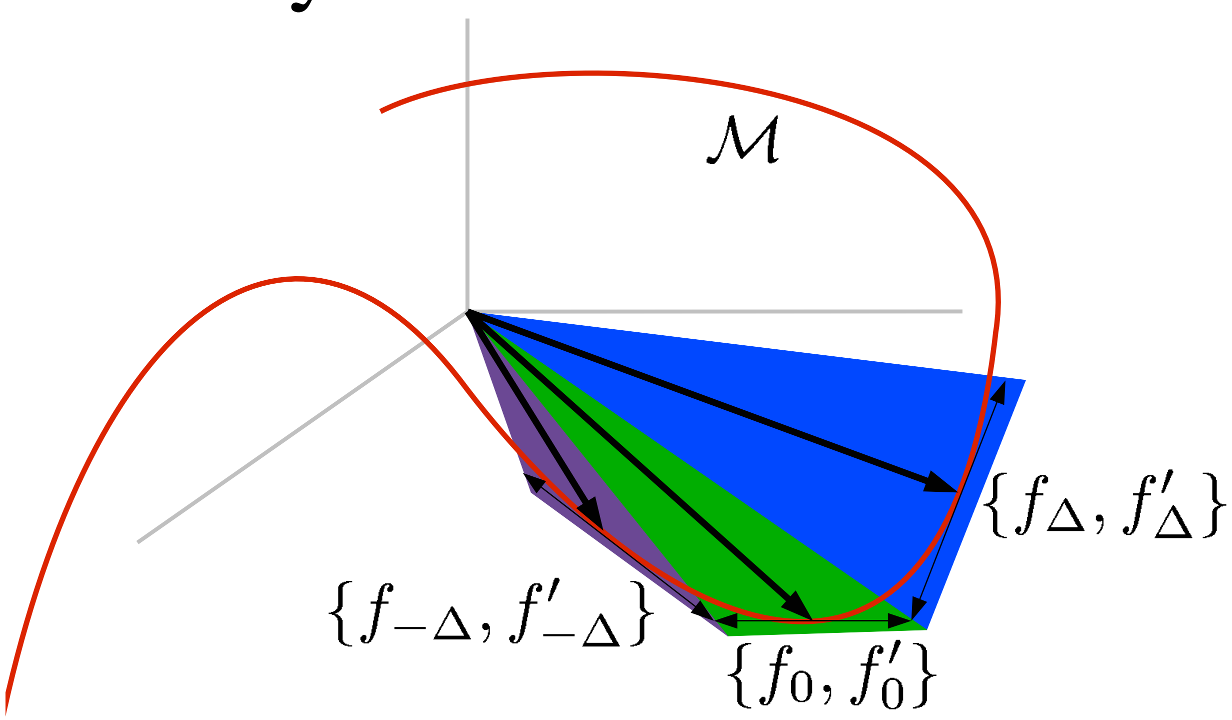
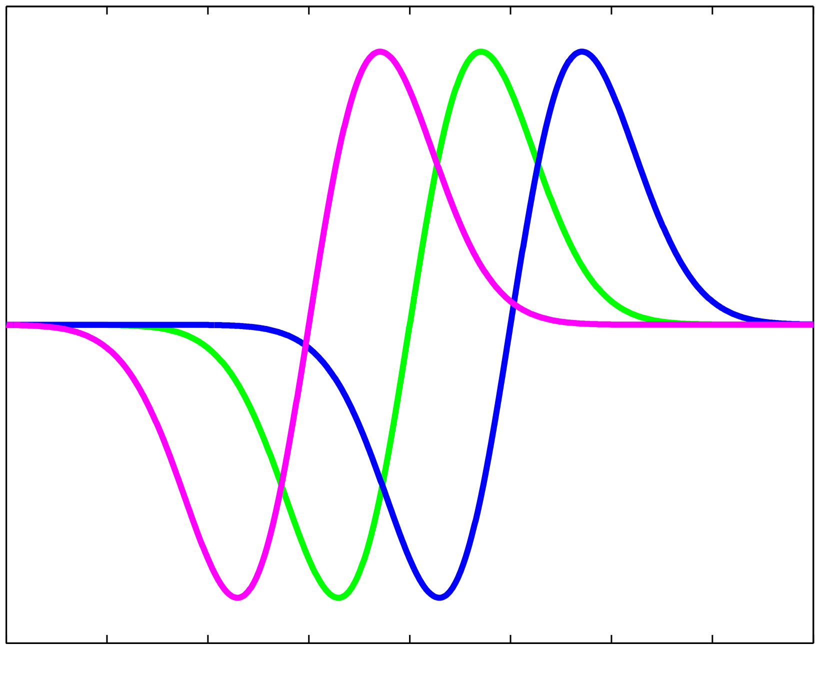


CBP - Taylor1





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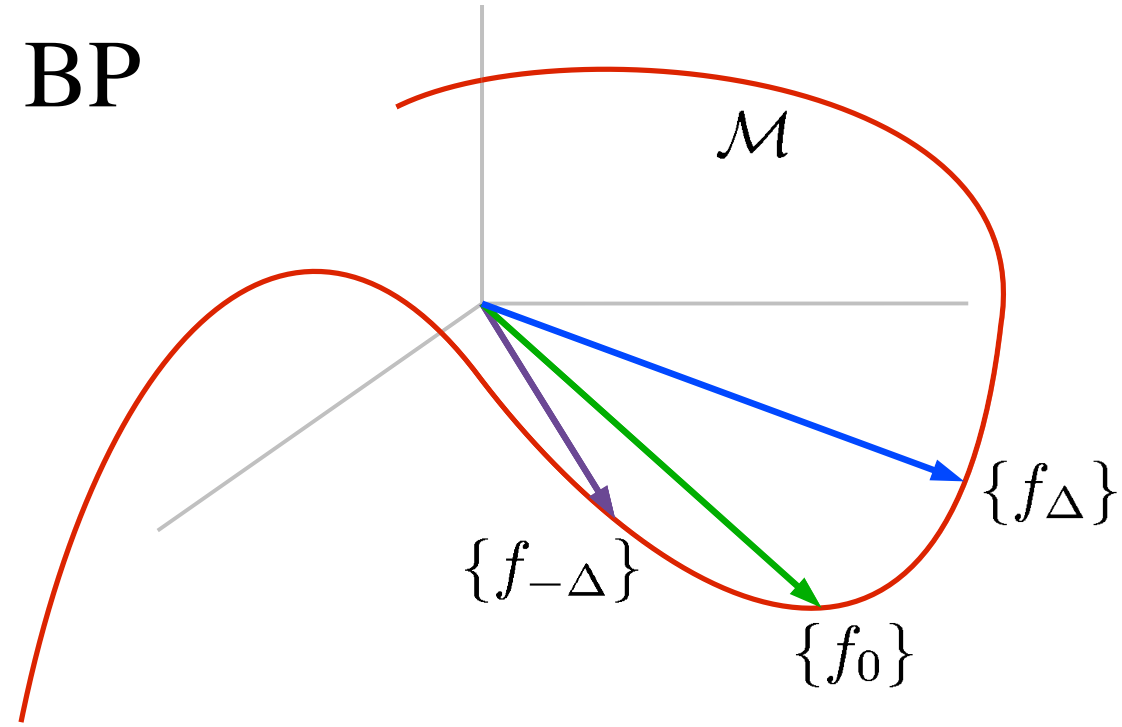

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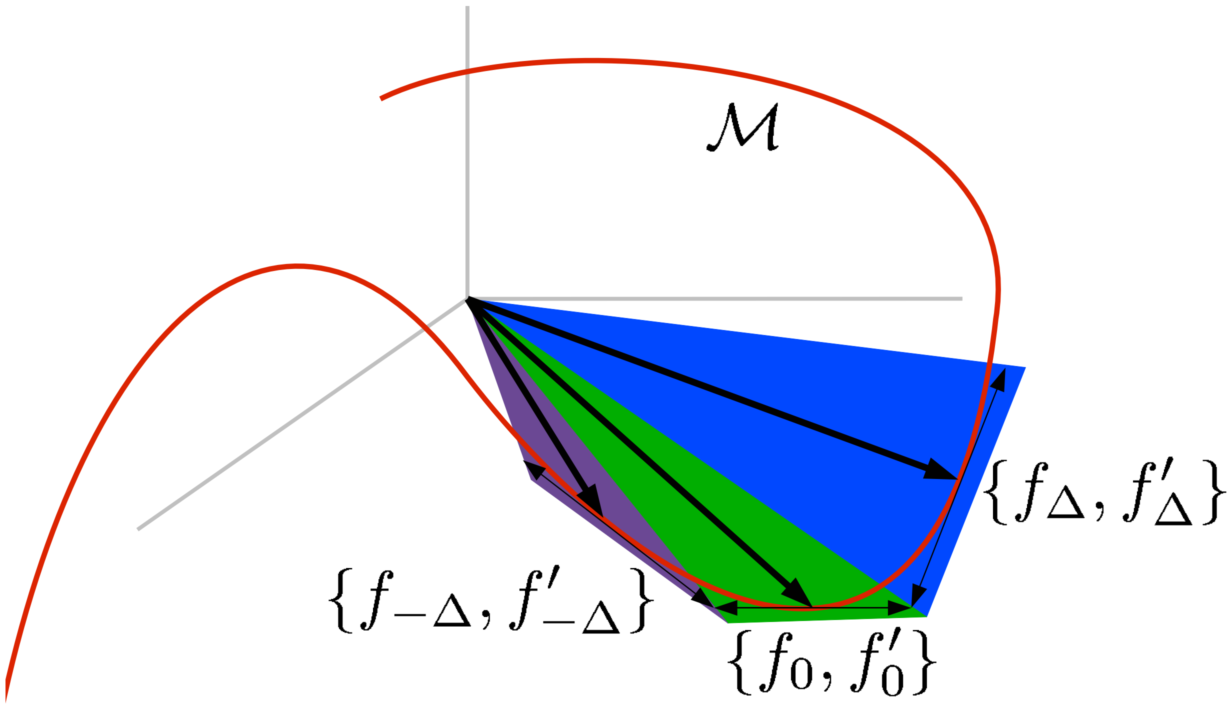
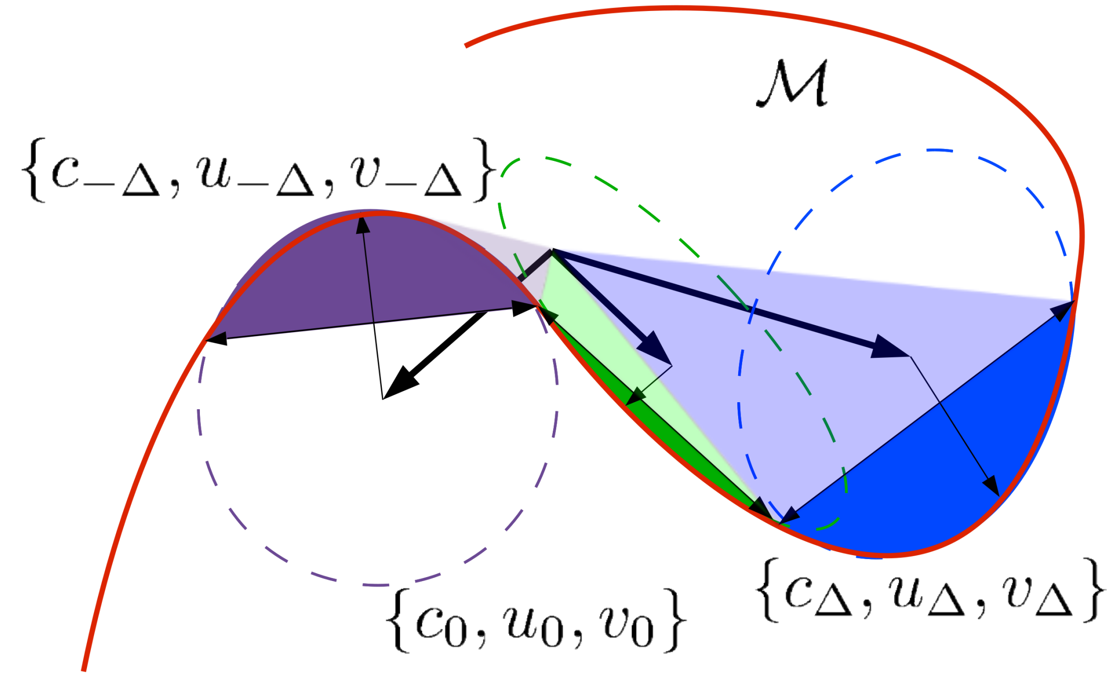
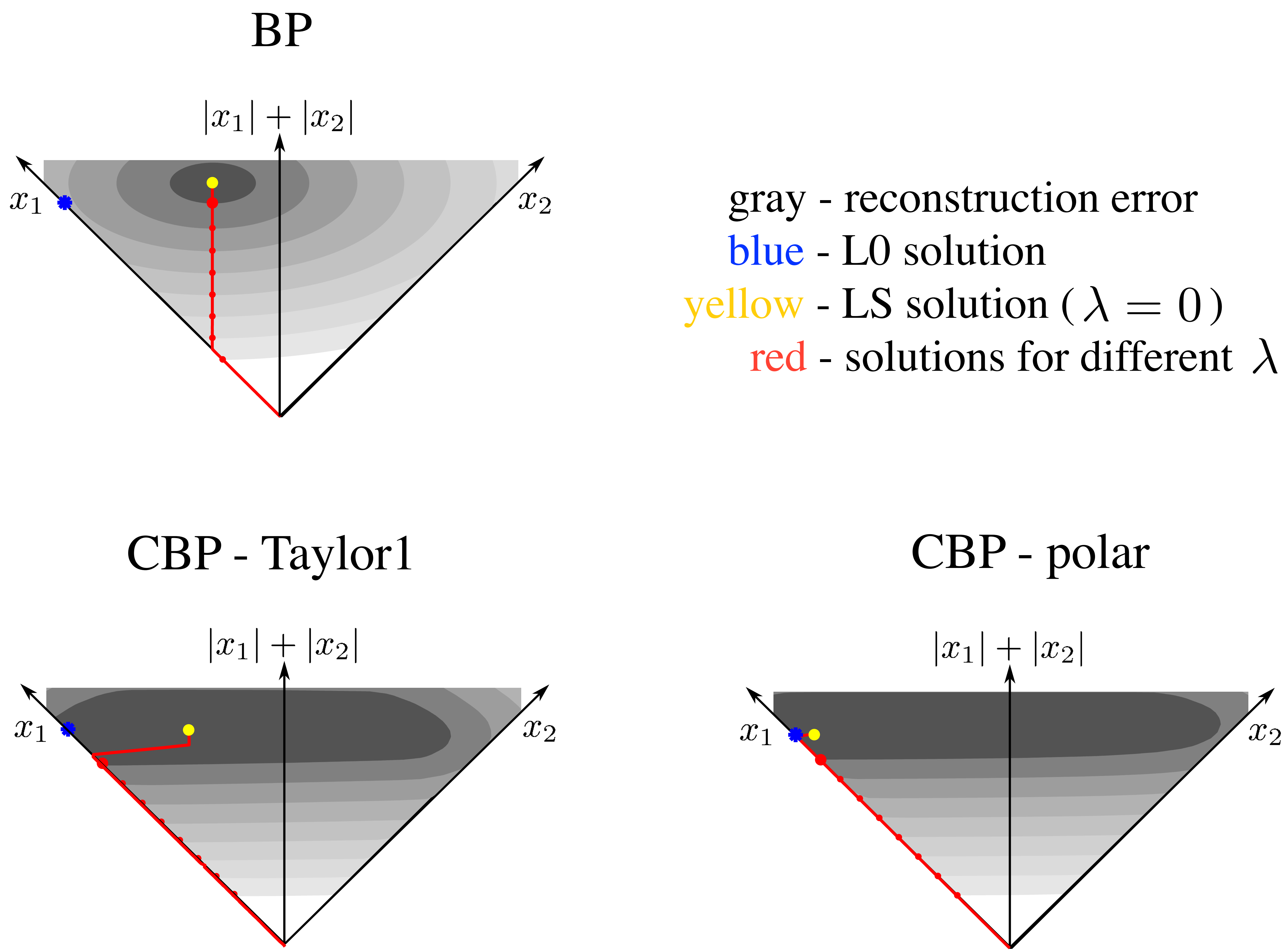


CBP - polar

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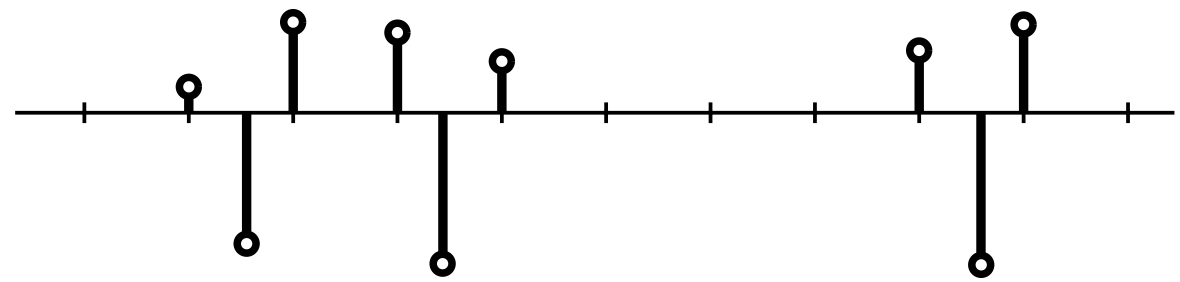

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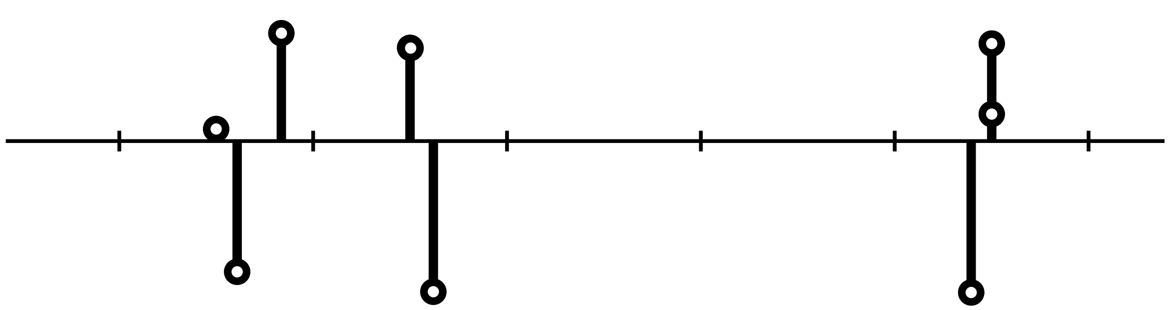

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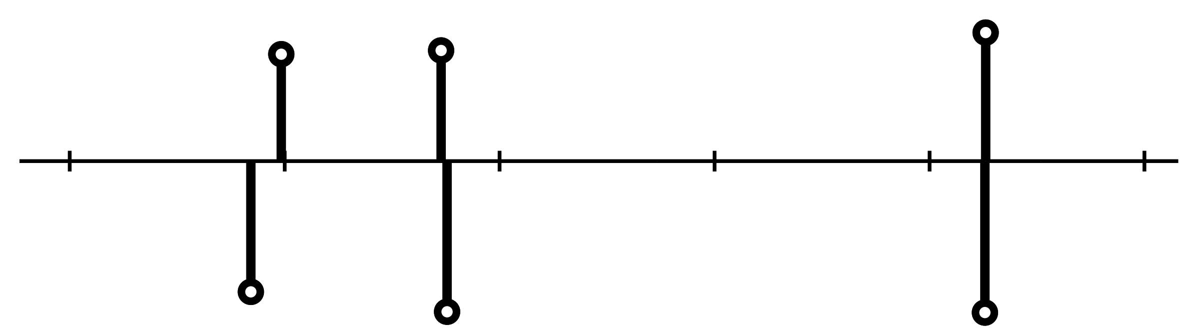

upward: estimated locations/amplitudes, with optimized $\{\lambda, \Delta\}$
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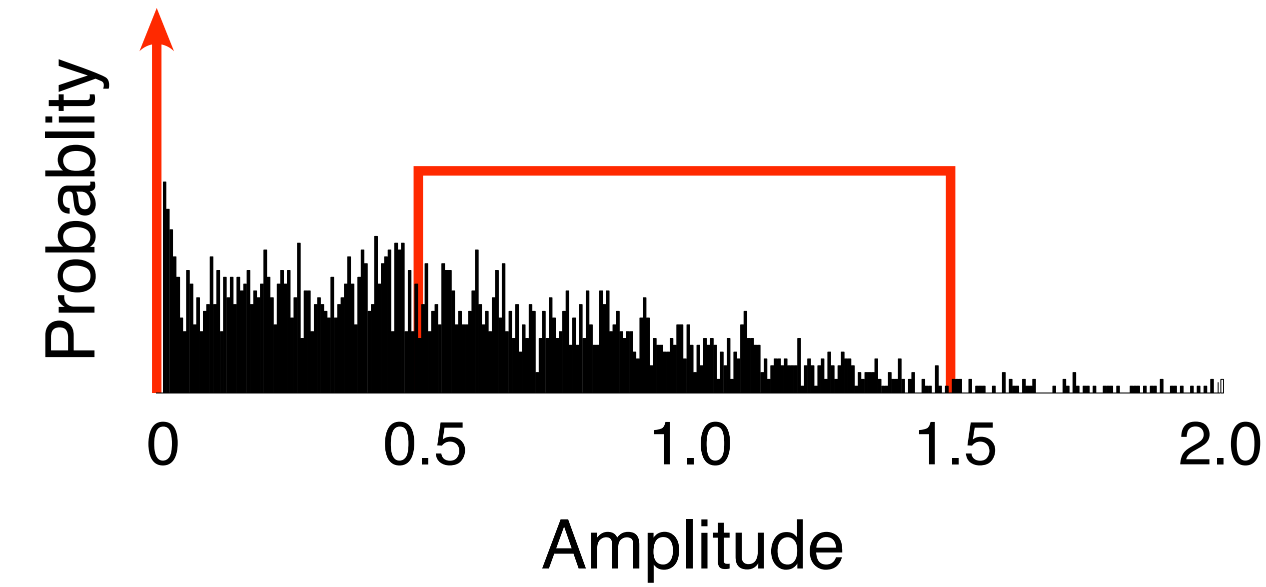

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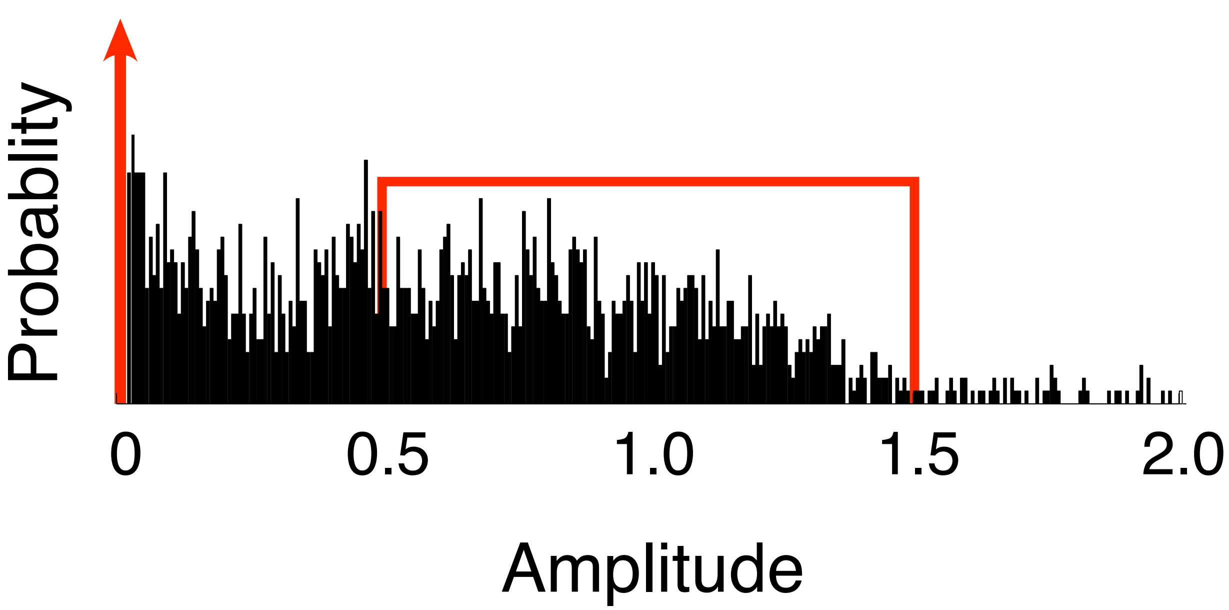

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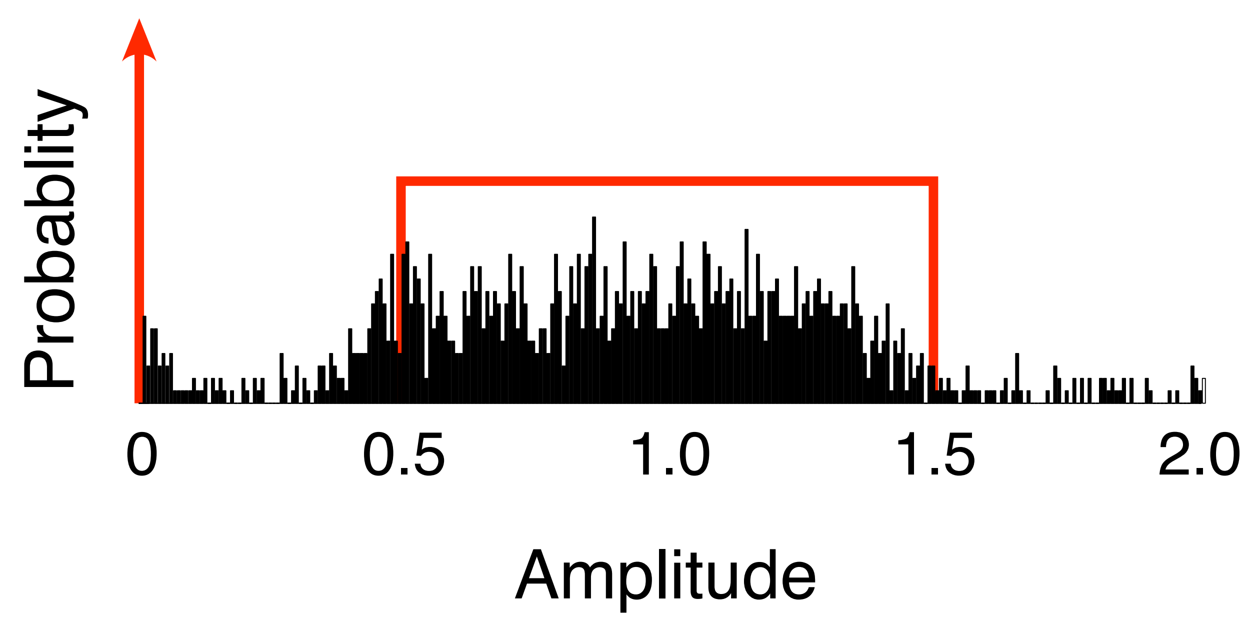
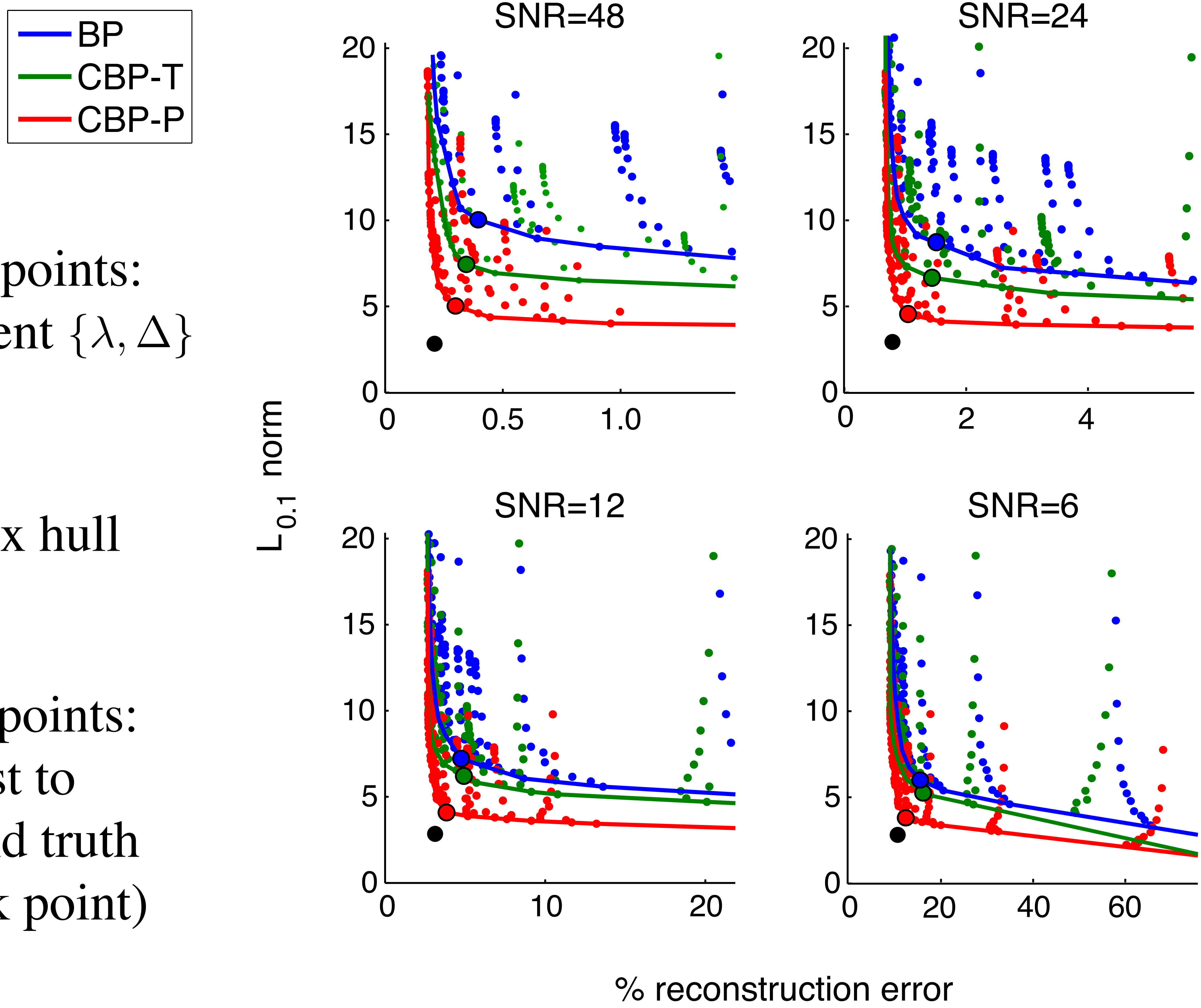


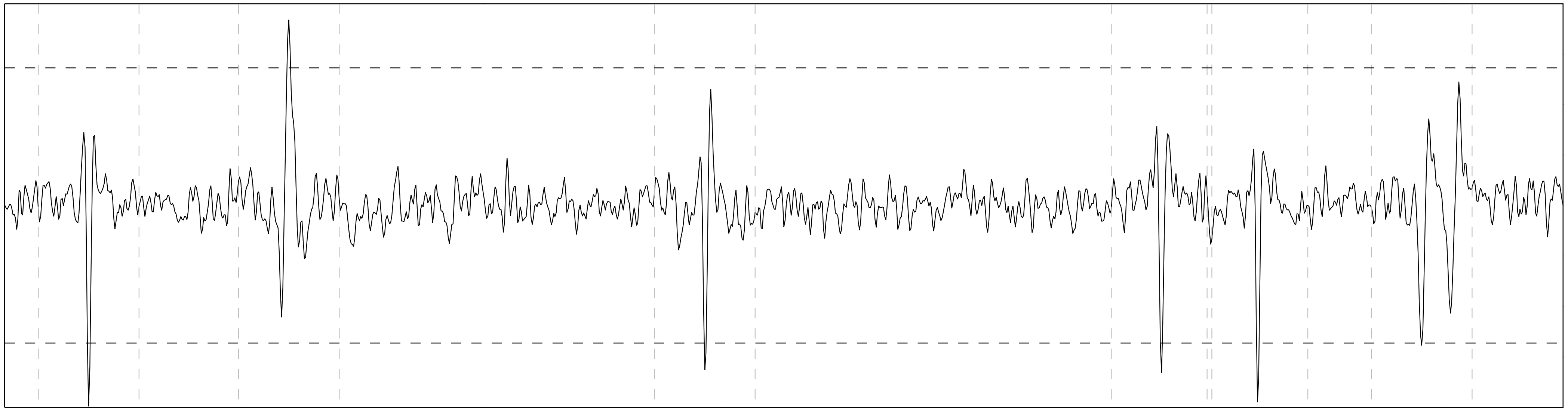
small points: different $\{\lambda, \Delta\}$

lines: convex hull

large points: closest to ground truth (black point)



Application to neural “spike sorting”



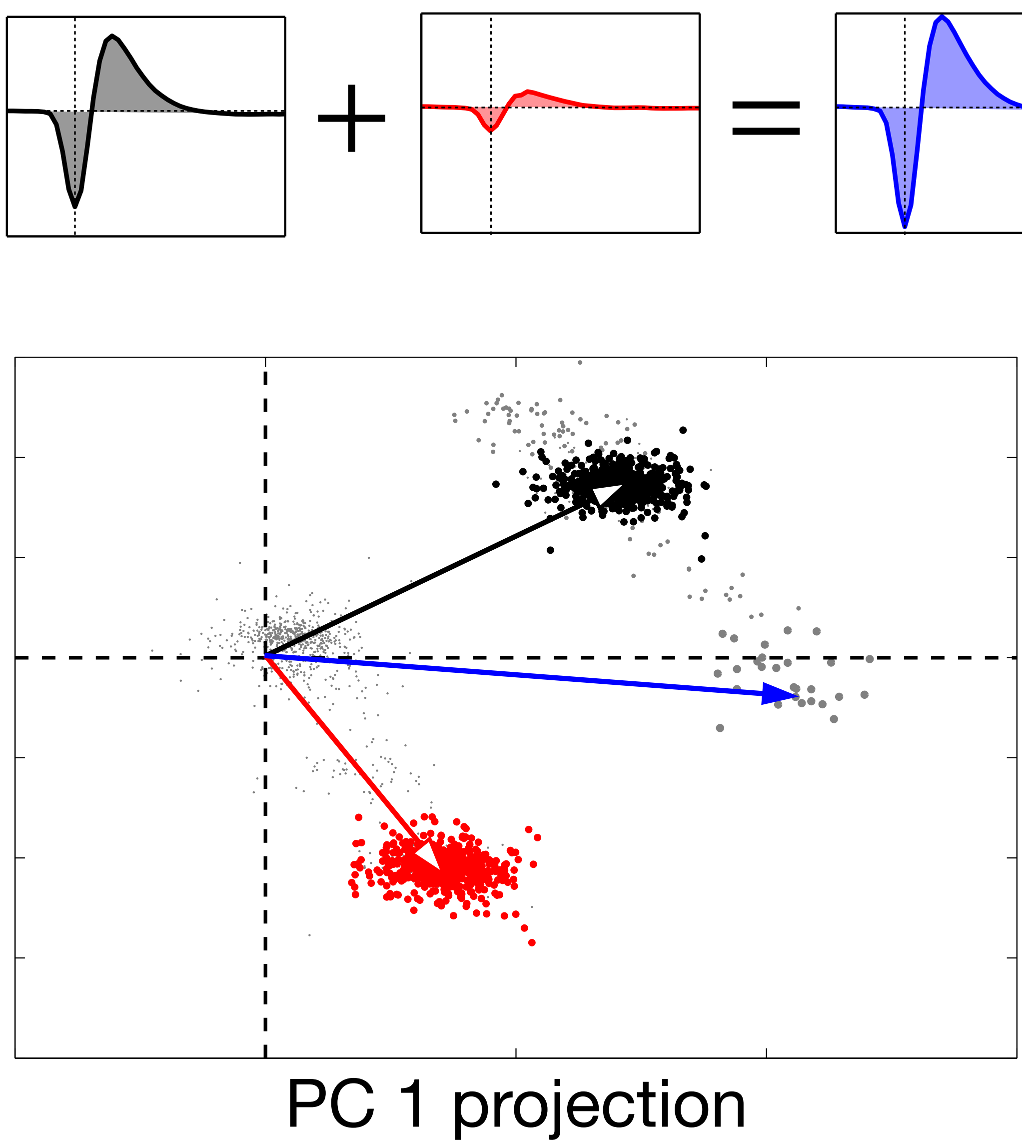
Widely-used solution:

1. Threshold to find segments containing spikes
2. Reduce dimensionality of segments using PCA
3. Identify spikes using clustering (e.g., K-means)

Guaranteed failure for overlapping spikes!

Failures of clustering for near-synchronous spikes

synchronous spiking



[Pillow et. al. 2013]

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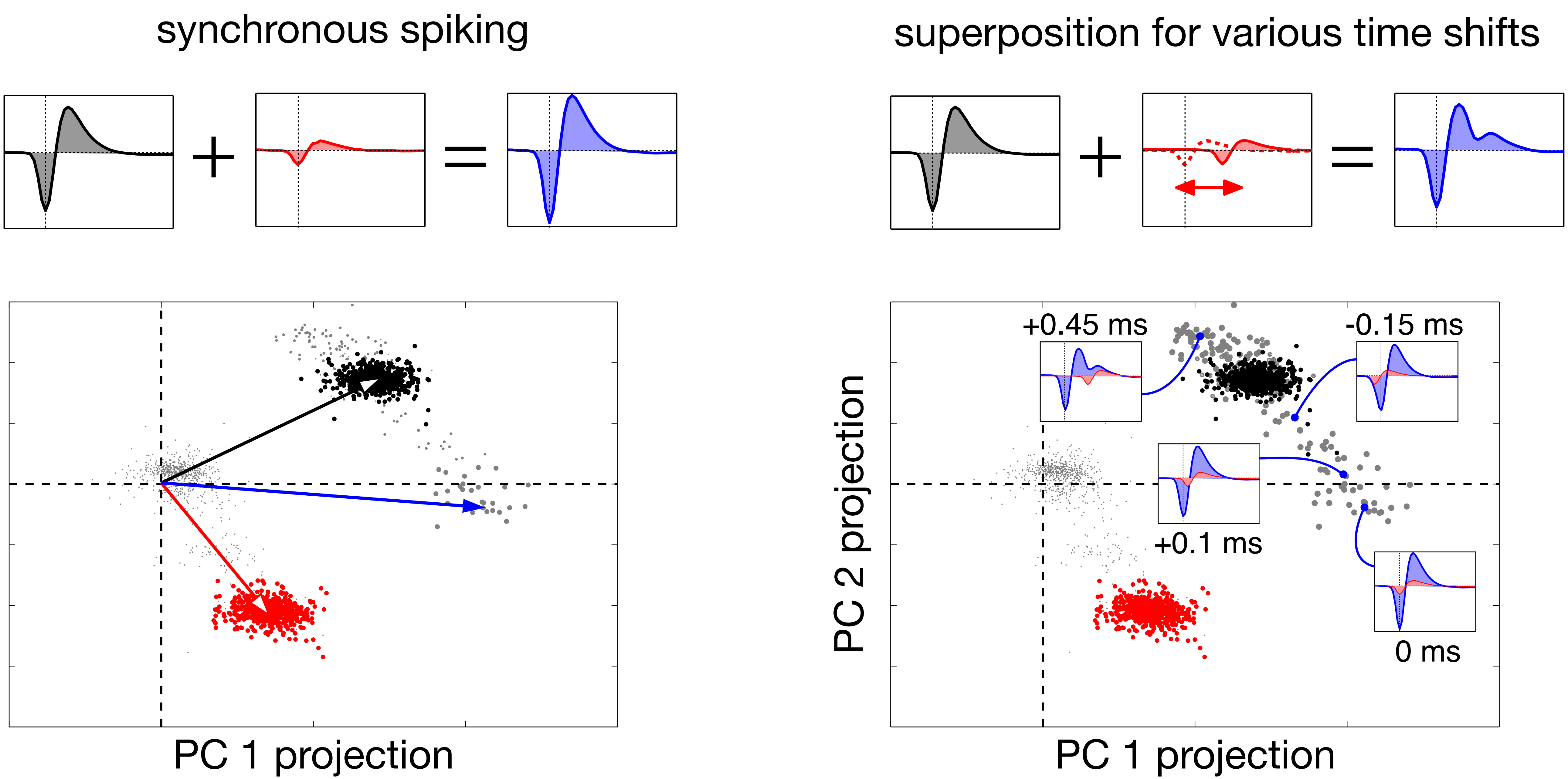

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Initialize f (random, or use K-means cluster centers)

[Ekanadham et al, 2013]

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Use CBP to solve for $\{x_n, w_n, z_n\}$:

$$\arg\min_{\vec{x},\vec{w},\vec{z}} ||y(t) - \sum_{n=1}^{N} x_n c(t-n\Delta) + w_n u(t-n\Delta) + z_n v(t-n\Delta)||_2^2 + \lambda||\vec{x}||_1$$

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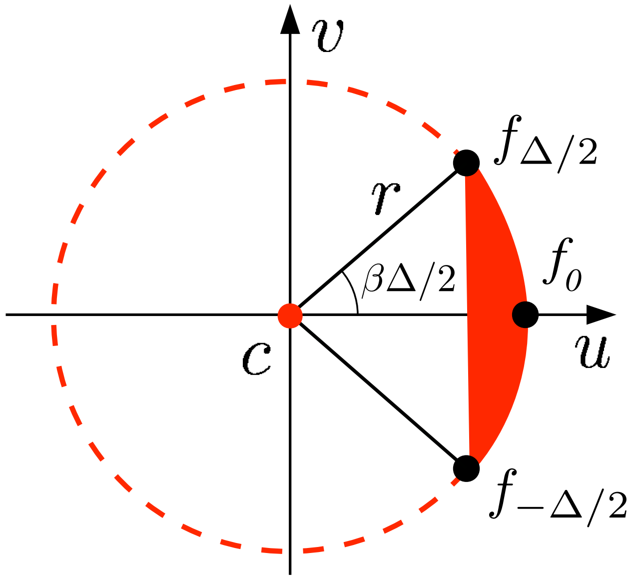
Convert $\{x_n, w_n, z_n\}$ back to $\{a_j, \tau_j\}$



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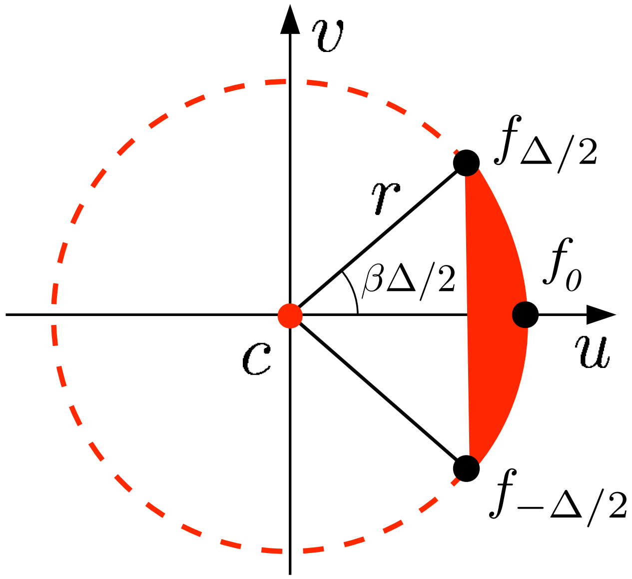

Solve for f (least squares):

$$\arg\min_{f} ||y(t) - \sum_{j} a_j f(t - \tau_j)||_2^2$$

[Ekanadham et al, 2013]

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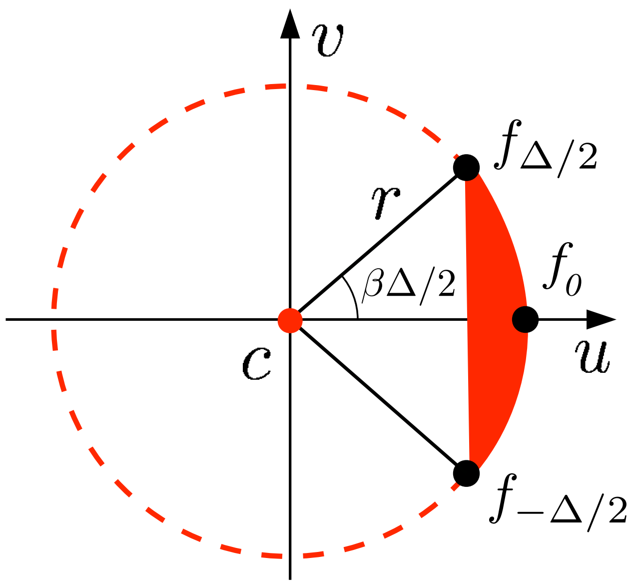

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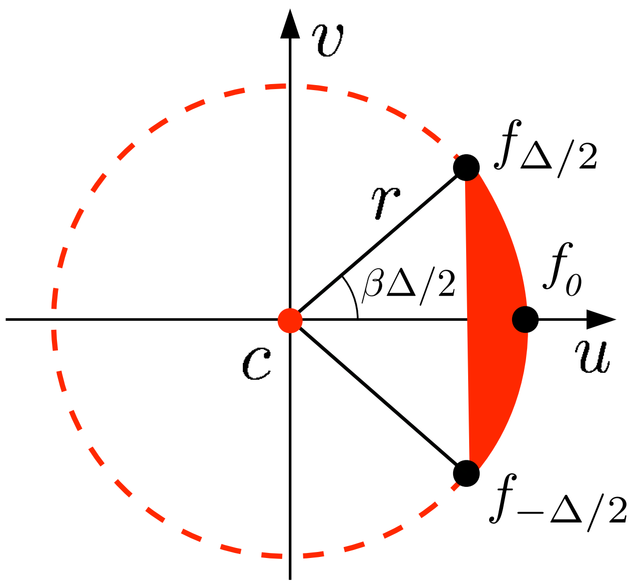


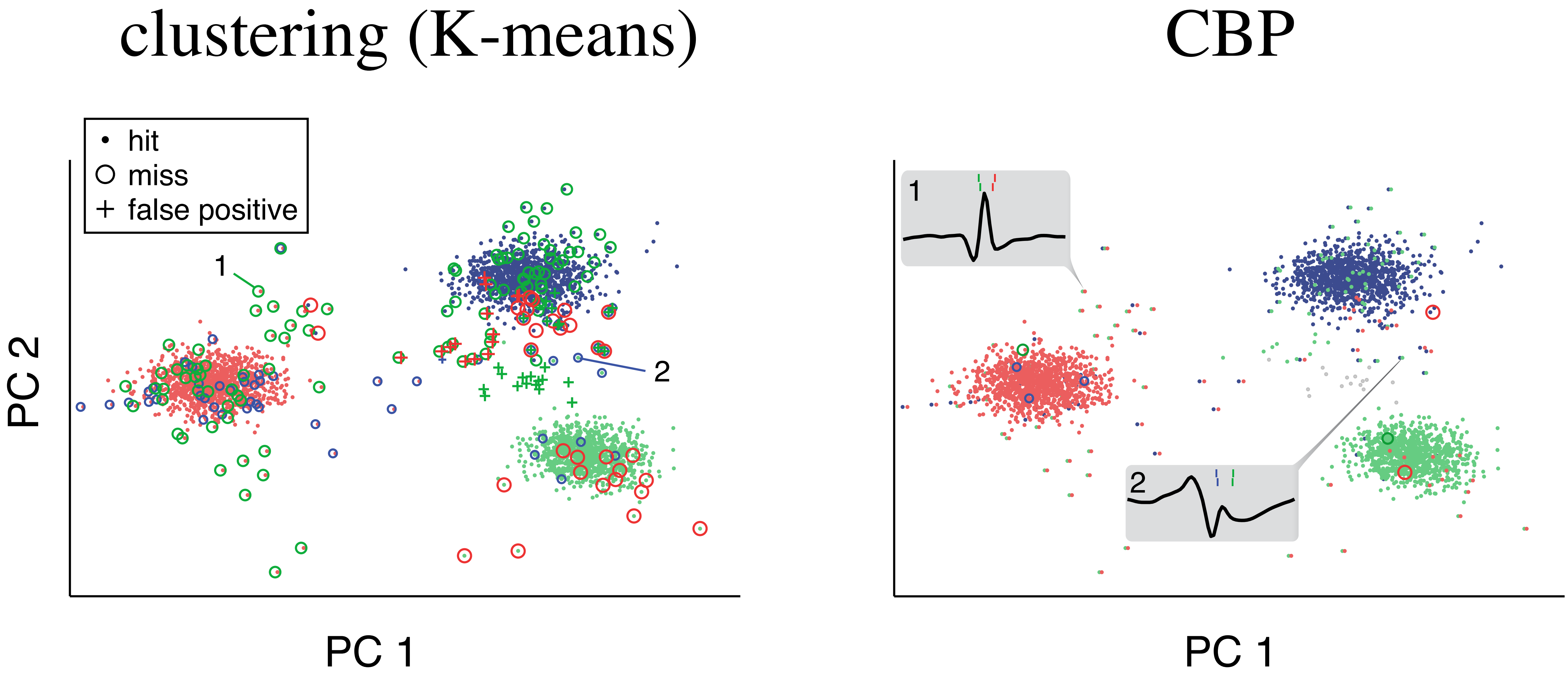
Solve for f (least squares):

$$\arg\min_{f} ||y(t) - \sum_{j} a_j f(t-\tau_j)||_2^2$$

Binarize to obtain final spike estimates

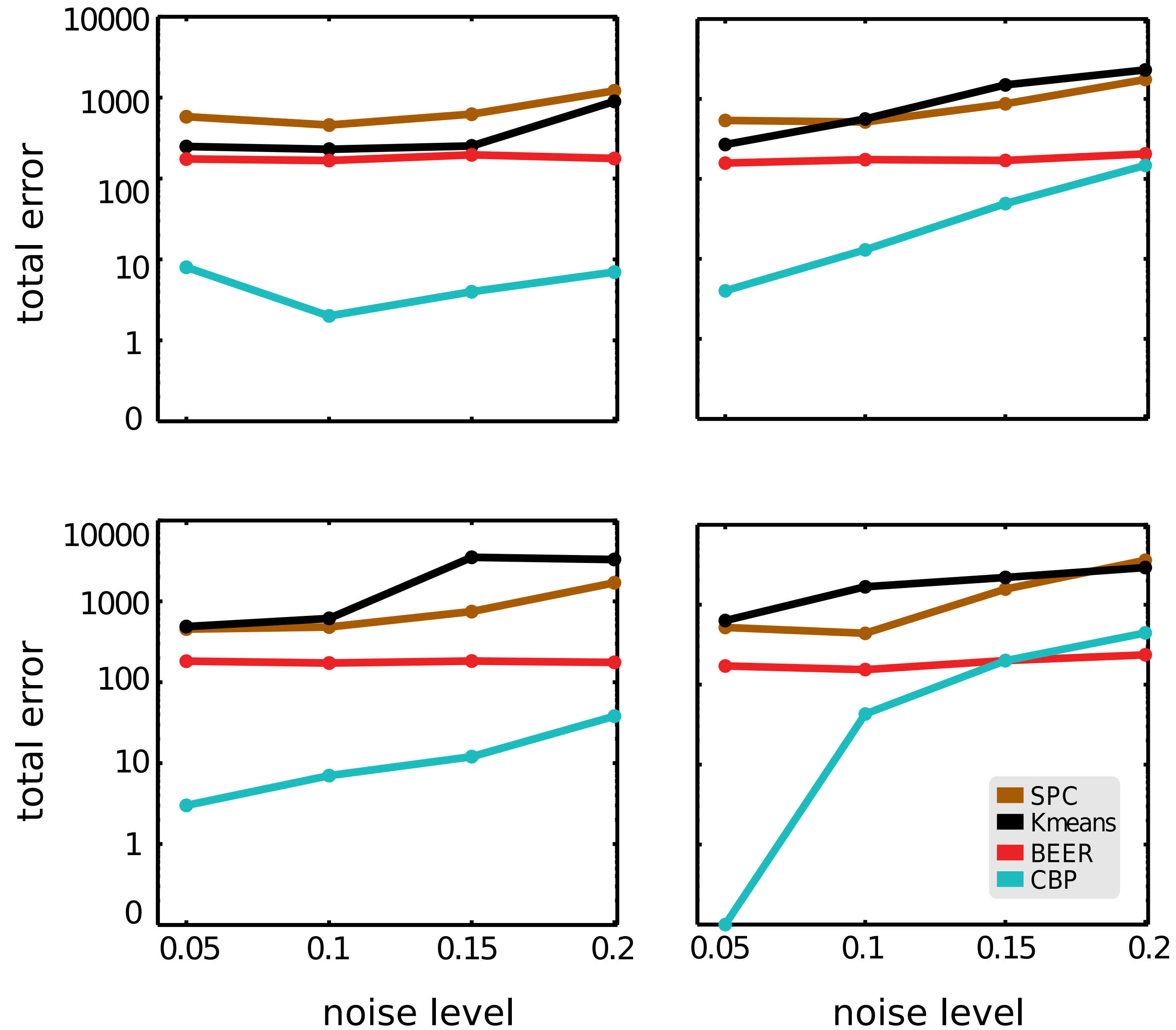
[Ekanadham et al, 2013]

Simulated data [Quiroga et. al. 2004]



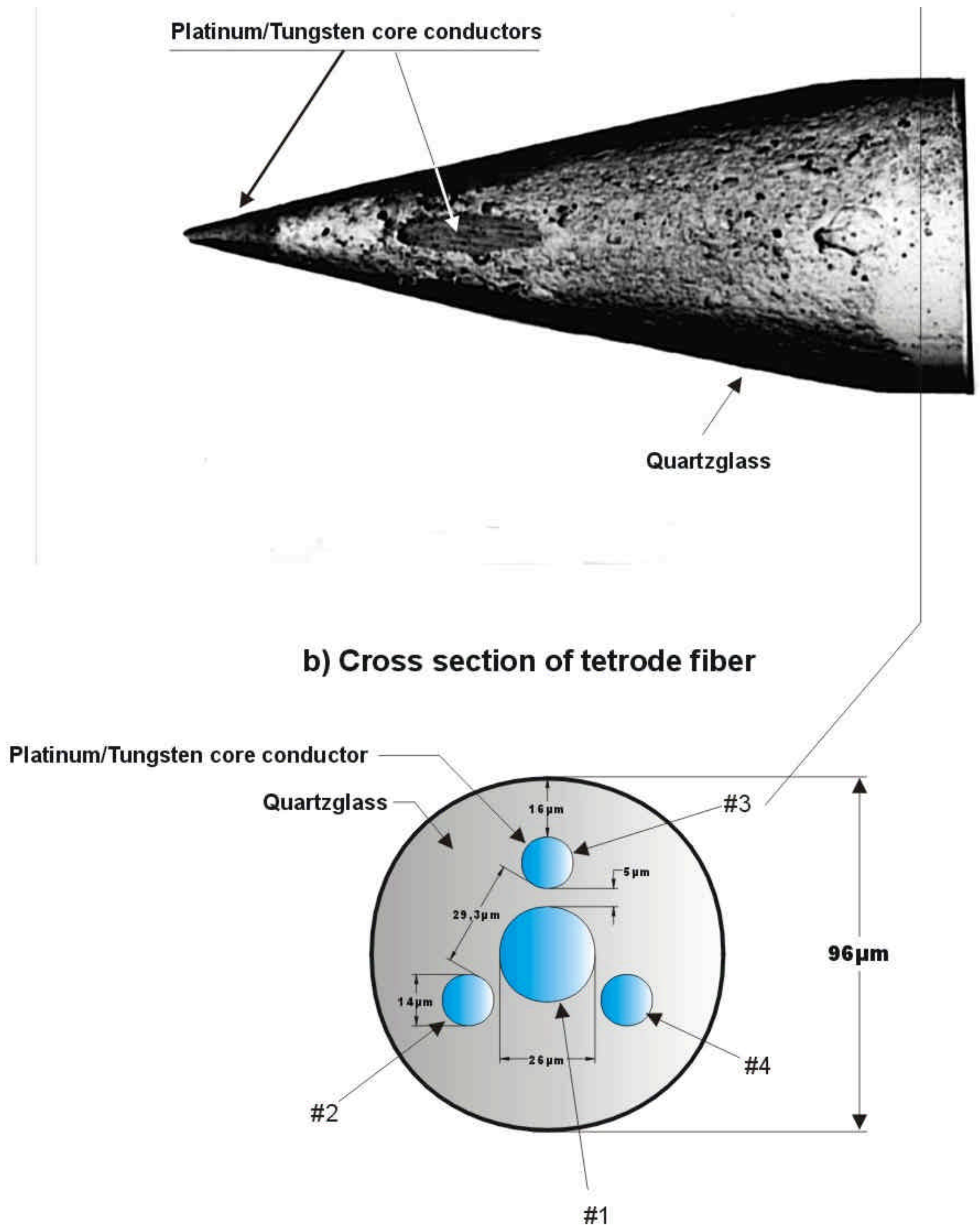
[Ekanadham et al, 2013]

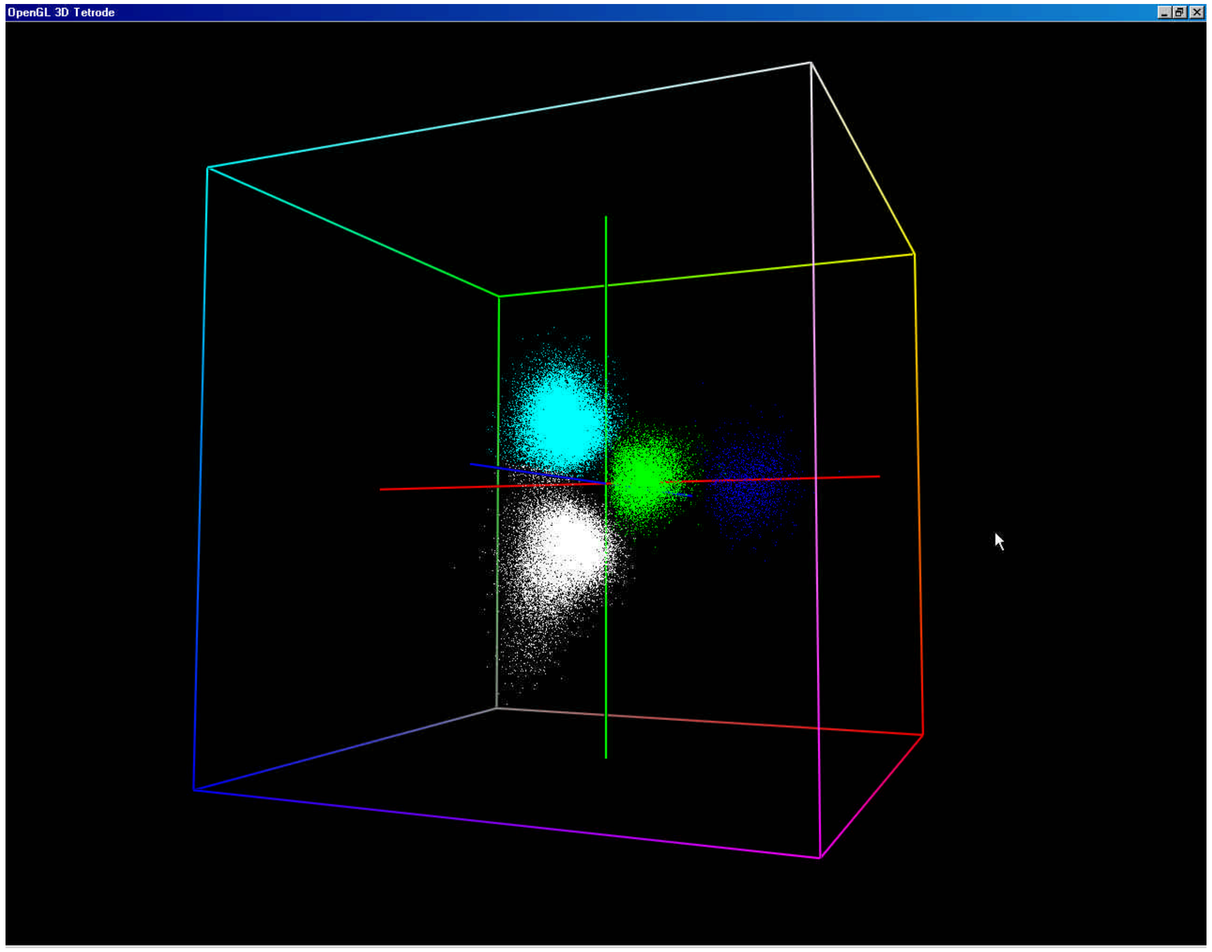
Simulated data (4 data sets) [Quiroga et. al. 2004]



SPC = super-paramagnetic clustering [Quiroga et. al. 2004]

BEER = “best ellipsoid error rate” - elliptical clustering, trained on ground truth data [Harris et. al. 2000]



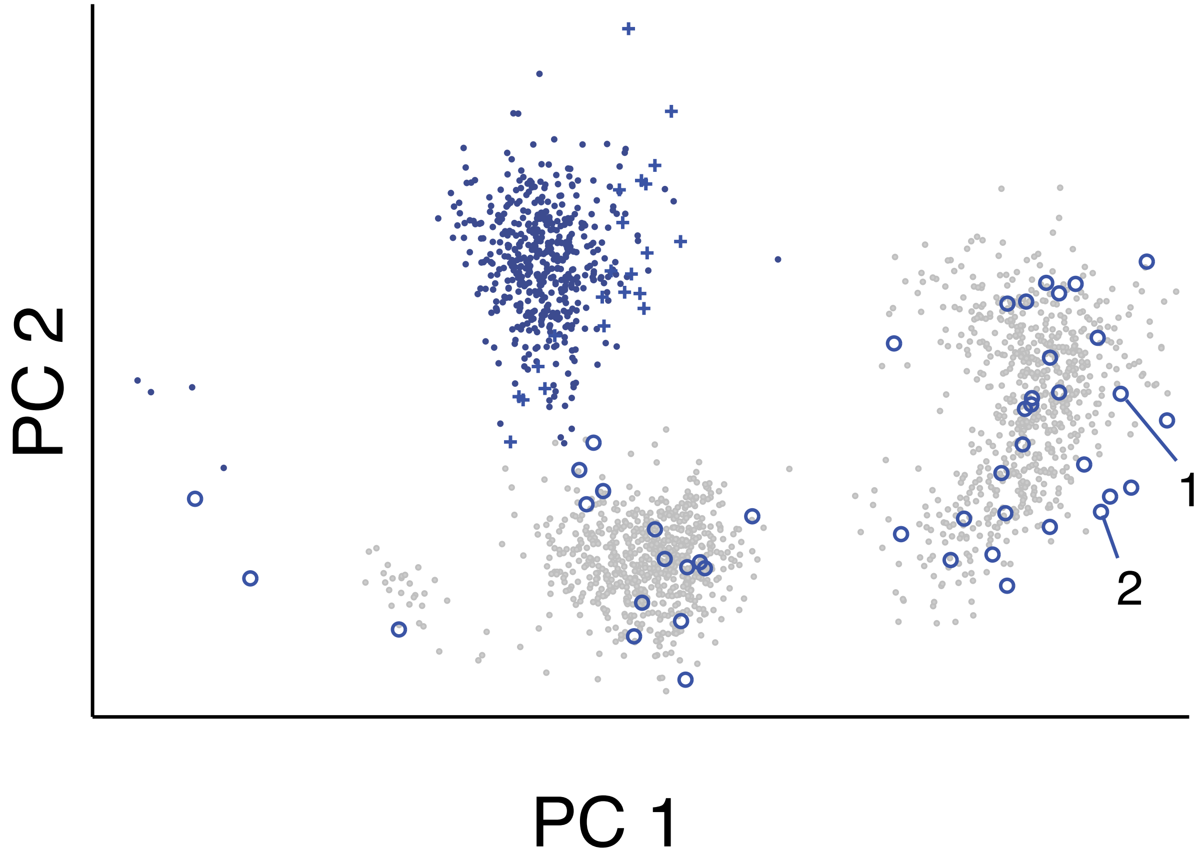


Tetrode electrode assembly

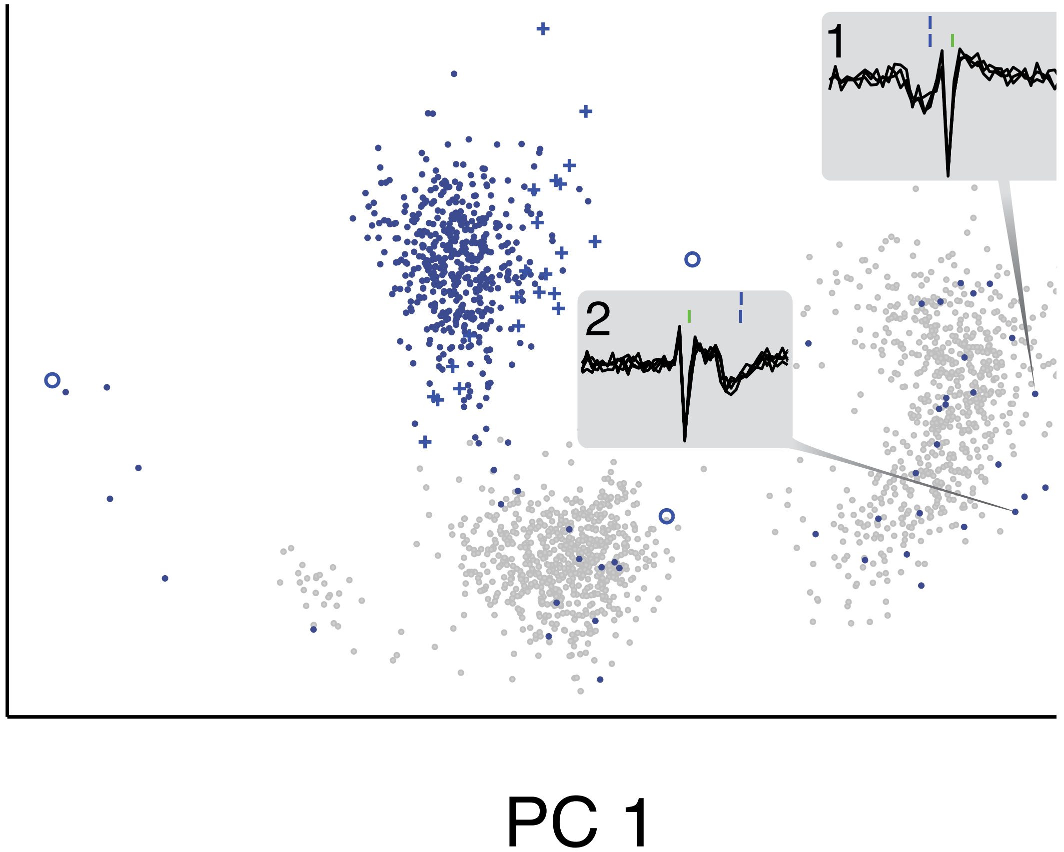
Thomson Recording GmbH, Giessen Germany

Tetrode data, with intracellular ground truth (for one cell) [Harris et. al., 2000]

clustering (K-means)



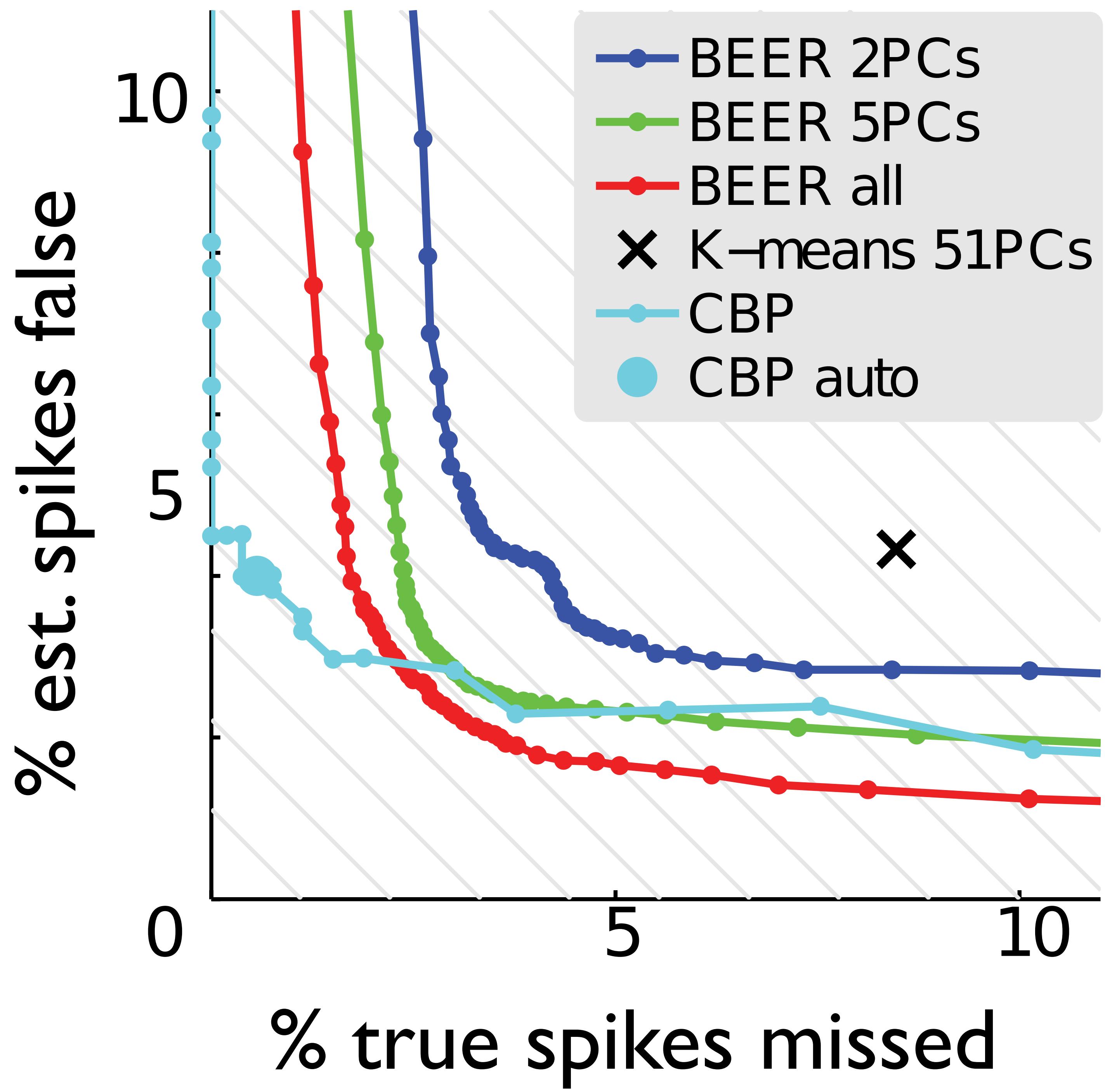
CBP



[Ekanadham et al, 2013]

Tetrode data, intracellular ground truth (for one cell)

[Harris et. al., 2000]



[Ekanadham et al, 2013]

Conclusions

- CBP: Sparse signal decomposition with continuously shifted feature waveforms
  - Translation manifold approximated with circular arcs
  - Can significantly outperform standard BP/LASSO
  - State-of-the-art results in neural spike identification
- Extensions:
  - Priors on coefficient amplitudes
  - Proximal methods / ADMM
  - Other transformations (dilation, rotation, modulation)