Primal-dual fixed point algorithms for separable minimization problems and their applications in imaging

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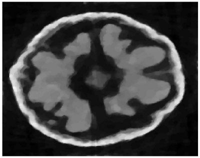
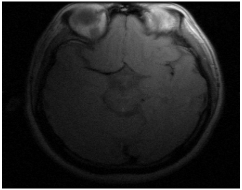




Outline

Background

Primal dual fixed point algorithm

Extensions

Conclusions

Background

General convex separable minimization model

$$x^* = \underset{x \in \mathbb{R}^n}{\arg\min} \quad (f_1 \circ B)(x) + f_2(x)$$

- $B : \mathbb{R}^n \to \mathbb{R}^m$ a linear transform.
- f_1, f_2 are proper l.s.c convex function defined in a Hilbert space.
- f_2 is differentiable on $\mathbb{R}^n$ with a $1/\beta$-Lipschitz continuous gradient for some $\beta \in (0, +\infty)$

Operator splitting methods

$$\min_{x \in \mathcal{X}} \quad f(x) + h(x)$$

where $f,\ h$ are proper l.s.c. convex and h is differentiable on $\mathcal{X}$ with a $1/\beta$-Lipschitz continuous gradient

- Define proximal operator prox_f as

$$\begin{array}{rccl} \mathrm{prox}_f : & \mathcal{X} & \rightarrow & \mathcal{X} \\ & x & \mapsto & \arg\min_{y \in \mathcal{X}} f(y) + \frac{1}{2}\|x - y\|_2^2, \end{array}$$

- Proximal forward-Backward splitting (PFBS)[1]

$$x_{k+1} = \mathrm{prox}_{\gamma f}(x_k - \gamma \nabla h(x_k)),$$

  for $0 < \gamma < 2\beta$,

- Many more other variants and related work (partial list: ISTA, FPCA, FISTA,GFB...).

An important assumption: $\mathrm{prox}_f(x)$ is an easy problem !!

[1] Moreau 1962; Lions-Mercier; 1979, Combettes- Wajs, 2005; FPC

Soft shrinkage

For $f(x) = \mu\|x\|_1$,

$$\mathsf{prox}_f(x) = \mathsf{sign}(x) \cdot \max(|x| - \mu, \mathbf{0})$$

- Componentwise thus efficient.
- Generally no easy form for $\mathsf{prox}_{\|Bx\|_1}(x)$ for non-invertible B.
- Largely used in compressive sensing and imaging sciences. Similar formula for matrix nuclear norm minimization (restore low rank matrix) or other matrix sparsity.

Methods on splitting form

$$(SPP) \qquad \max_{y} \inf_{x,z} \quad \{f_1(z) + f_2(x) + \langle y, Bx - z\rangle\}$$

Augmented Lagrangian:

$$L^\nu(x, z;\ y) = f_1(z) + f_2(x) + \langle y, Bx - z\rangle + \frac{\nu}{2}\|Bx - z\|^2$$

- Alternating direction of multiplier method (ADMM) (Glowinski et al. 75, Gabay et al. 83)

$$\begin{cases} x^{k+1} &= \arg\min_x L^\nu(x, z^k; y^k) \\ z^{k+1} &= \arg\min_z L^\nu(x^{k+1}, z; y^k) \\ y^{k+1} &= y^k + \gamma\nu(Bx^{k+1} - z^{k+1}) \end{cases}$$

- Split Inexact Uzawa (SIU/BOS) Method[2]

$$\begin{cases} x^{k+1} &= \arg\min_u L^\nu(x, z^k; y^k) + \|x - x^k\|^2_{D_1} \\ z^{k+1} &= \arg\min_z L^\nu(x^{k+1}, z; y^k) + \|z - z^k\|^2_{D_2} \\ Cy^{k+1} &= Cy^k + (Bx^{k+1} - z^{k+1}) \end{cases}$$

[2] Zhang-Burger-Osher,2011

Methods for Primal-Dual from (PD)

$$(PD) \qquad : \min_x \sup_y -f_1^*(y) + \langle y, Bu \rangle + f_2(x)$$

- Primal-dual hybrid Gradient (PDHG) Method (Zhu-Chan, 2008)

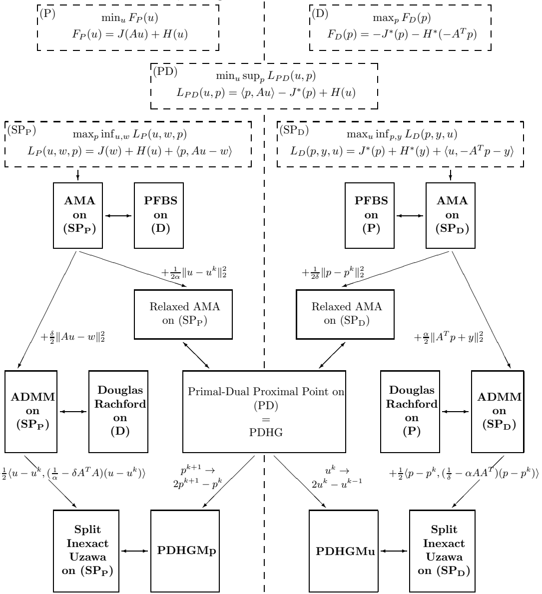
$$y^{k+1} = \arg\max_y -f_1^*(y) + \langle y, Bx^k \rangle - \frac{1}{2\delta_k}\|y - y^k\|_2^2$$
$$x^{k+1} = \arg\min_x f_2(x) + \langle B^T y^{k+1}, x \rangle + \frac{1}{2\alpha_k}\|x - x^k\|_2^2$$

- Modified PDHG (PDHGMp, Esser-Zhang-Chan,2010, equivalent to SIU on (SPP); Pock-Cremers-Bischof-Chambolle, 2009, Chambolle-Pock, 2011($\theta = 1$))
Replace p^k in first step of PDHG with $2p^k - p^{k-1}$ to get PDHGMp:

$$x^{k+1} = \arg\min_x f_2(x) + \left\langle B^T \left(2y^k - y^{k-1}\right), x \right\rangle + \frac{1}{2\alpha}\|x - x^k\|_2^2$$
$$y^{k+1} = \arg\min_y f_1^*(y) - \langle y, Bx^{k+1} \rangle + \frac{1}{2\delta}\|y - y^k\|_2^2$$

Connections[3]

14 E. Esser, X. Zhang and T.F. Chan



Legend:
(P): Primal
(D): Dual
(PD): Primal-Dual
($\mathrm{SP_P}$): Split Primal
($\mathrm{SP_D}$): Split Dual

AMA: Alternating Minimization Algorithm (4.2.1)
PFBS: Proximal Forward Backward Splitting (4.2.1)
ADMM: Alternating Direction Method of Multipliers (4.2.2)
PDHG: Primal Dual Hybrid Gradient (4.2)
PDHGM: Modified PDHG (4.2.3)
Bold: Well Understood Convergence Properties

First order methods in imaging sciences

- Many efficient algorithms exist: "augmented lagrangian", "splitting methods","alternating methods", "primal-dual", "fixed point methods" etc.
- Huge number of seemingly related methods.
- Many of them requires subproblem solving involving inner-iterations, ad-hoc parameter selections, which can not be clearly controlled in real implementation.
- Need for methods with simple, explicit iterations capable of solving large scale, often non-differentiable convex models with separable structure
- Convergence analysis are mainly for the objective function, or in ergodic sense. Most of them have sublinear convergence ($O(1/N)$ or $O(1/N^2)$).

Primal dual fixed point methods

Fixed point algorithm based on proximity operator (FP^{2}O)

For a given $b \in \mathbb{R}^n$, solve for

$$\mathrm{prox}_{f_1 \circ B}(b)$$

$$H(v) = (I - \mathrm{prox}_{\frac{f_1}{\lambda}})(Bb + (I - \lambda BB^T)v) \quad \text{for all } v \in \mathbb{R}^m$$

FP^{2}O (Micchelli-Shen-Xu, 11')

Step 1: Set $v_0 \in \mathbb{R}^m$, $0 < \lambda < 2/\lambda_{max}(BB^T)$, $\kappa \in (0, 1)$.
Step 2: Calculate v^*, which is the fixed point of H, with iteration

$$v_{k+1} = H_\kappa(v_k)$$

where $H_\kappa = \kappa I + (1 - \kappa)H$ for $\kappa \in (0, 1)$
Step 3: $\mathrm{prox}_{f_1 \circ B}(b) = b - \lambda B^T v^*$.

Solve for general problem

Solve for

$$x^* = \operatorname*{arg\,min}_{x \in \mathbb{R}^n} \quad (f_1 \circ B)(x) + f_2(x)$$

PFBS_FP^{2}O (Argyriou-Micchelli-Pontil-Shen-Xu,11')

Step 1: Set $x_0 \in \mathbb{R}^n$, $0 < \gamma < 2\beta$.
Step 2: for $k = 0, 1, 2, \cdots$
Calculate $x_{k+1} = \text{prox}_{\gamma f_1 \circ B}(x_k - \gamma \nabla f_2(x_k))$ using FP^{2}O.
end for

Note: Inner iterations are involved, no clear stopping criteria and error analysis!

Primal dual fixed point algorithm

Primal-dual fixed points algorithm based on proximity operator, PDFP^{2}O.

Step 1: Set $x_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^m$, $0 < \lambda \leq 1/\lambda_{max}(BB^T)$, $0 < \gamma < 2\beta$.

Step 2: for $k = 0, 1, 2, \cdots$

$$x_{k+1/2} = x_k - \gamma \nabla f_2(x_k),$$

$$v_{k+1} = (I - \text{prox}_{\frac{\gamma}{\lambda} f_1})(Bx_{k+1/2} + (I - \lambda BB^T)v_k),$$

$$x_{k+1} = x_{k+1/2} - \lambda B^T v_{k+1}.$$

end for

- No inner iterations if $\text{prox}_{\frac{\gamma}{\lambda} f_1}(x)$ is an easy problem.
- Extension of FP2^{2}O and PFBS.
- Can be extended to $\kappa-$ average fixed point iteration

Fixed point operator notion

Define $T_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ as

$$T_1(v,x) = (I - \mathrm{prox}_{\frac{\gamma}{\lambda} f_1})(B(x - \gamma\nabla f_2(x)) + (I - \lambda BB^T)v)$$

and $T_2 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$T_2(v,x) = x - \gamma\nabla f_2(x) - \lambda B^T \circ T_1.$$

Denote $T : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ as

$$T(v,x) = (T_1(v,x),\ T_2(v,x))\,.$$

Convergence of PDFP2O

Theorem

Let $\lambda > 0, \gamma > 0$. *Suppose that* x^* *is a solution of*

$$x^* = \operatorname*{arg\,min}_{x \in \mathbb{R}^n} \quad (f_1 \circ B)(x) + f_2(x)$$

if and only if there exists $v^* \in \mathbb{R}^m$ *such that* $u^* = (v^*, x^*)$ *is a fixed point of* T.

Theorem

Suppose $0 < \gamma < 2\beta$, $0 < \lambda \leq 1/\lambda_{\max}(BB^T)$ *and* $\kappa \in [0, 1)$. *Let* $u_k = (v_k, x_k)$ *be a sequence generated by PDFP*^{2}*O. Then* $\{u_k\}$ *converges to a fixed point of* T *and* $\{x_k\}$ *converges to a solution of problem*

$$x^* = \operatorname*{arg\,min}_{x \in \mathbb{R}^n} \quad (f_1 \circ B)(x) + f_2(x)$$

Convergence rate analysis

Condition

For $0 < \gamma < 2\beta$ *and* $0 < \lambda \le 1/\lambda_{\max}(BB^T)$, *there exist* η_1, $\eta_2 \in [0, 1)$ *such that* $\|I - \lambda BB^T\|_2 \le \eta_1^2$ *and*

$$\|g(x) - g(y)\|_2 \le \eta_2 \|x - y\|_2 \quad \text{for all } x,\ y \in \mathbb{R}^n,$$

where $g(x) = x - \gamma \nabla f_2(x)$.

Remarks

- If B has full row rank, f_2 is strongly convex, this condition can be satisfied.
- As a typical example, consider $f_2(x) = \frac{1}{2}\|Ax - b\|_2^2$ with $A^T A$ full rank.

Linear convergence rate

Theorem

Suppose the above condition holds true. Let $u_k = (v_k, x_k)$ *be a fixed point iteration sequence of operator* T. *Then the sequence* $\{u_k\}$ *must converge to the unique fixed point* $u^* = (v^*,\ x^*) \in \mathbb{R}^m \times \mathbb{R}^n$ *of* T, *with* x^* *the unique solution of the minimization problem. Furthermore,*

$$\|x_k - x^*\|_2 \leq \frac{c\theta^k}{1-\theta},$$

where $c = \|u_1 - u_0\|_\lambda$, $\theta = \kappa + (1-\kappa)\eta \in (0,1)$ *and* $\eta = \max\{\eta_1, \eta_2\}$.

- Let $B = I$, $\lambda = 1$, $f_2(x)$ strongly convex, then PFBS converges linearly.
- If B is full row rank, then PFP^{2}O converges linearly.
- Related work: linear Convergence of the ADMM methods (Luo 2012, Deng-Yin, CAM12-52, Goldstein -O'Donoghue -Setzer,2012)

Connection with other primal dual algorithms

Table : The comparison between CP ($\theta = 1$) and PDFP^{2}O.

	PDHGm/CP ($\theta = 1$)	PDFP^{2}O
Form	$\bar{v}_{k+1} = (I + \sigma\partial f_1^*)^{-1}(\bar{v}_k + \sigma B y_k)$ $x_{k+1} = (I + \tau\nabla f_2)^{-1}(x_k - \tau B^T \bar{v}_{k+1})$ $y_{k+1} = 2x_{k+1} - x_k$	$\bar{v}_{k+1} = (I + \frac{\lambda}{\gamma}\partial f_1^*)^{-1}(\frac{\lambda}{\gamma}\bar{v}_k + B y_k)$ $x_{k+1} = x_k - \gamma\nabla f_2(x_k) - \gamma B^T \bar{v}_{k+1}$ $y_{k+1} = x_{k+1} - \gamma\nabla f_2(x_{k+1}) - \gamma B^T \bar{v}_{k+1}$
Condition	$0 < \sigma\tau < \frac{1}{\lambda_{\max}(BB^T)}$	$0 < \gamma < 2\beta$, $0 < \lambda \leq \frac{1}{\lambda_{\max}(BB^T)}$
Relation	$\sigma = \lambda/\gamma$, $\tau = \gamma$	

Connection with Splitting types of method

$$\min_{x\in\mathbb{R}^n} \quad \mu\|Bx\|_1 + \frac{1}{2}\|Ax-b\|_2^2$$

$$(\text{ASB}^4) \qquad \begin{cases} x_{k+1} = (A^TA+\nu B^TB)^{-1}(A^Tb+\nu B^T(d_k-v_k)), & \text{(1a)} \\ d_{k+1} = \text{prox}_{\frac{1}{\nu}f_1}(Bx_{k+1}+v_k), \\ v_{k+1} = v_k - (d_{k+1}-Bx_{k+1}), \end{cases}$$

for $\nu > 0$.

- SIU [5]: (1a) is replaced by

$$x_{k+1} = x_k - \delta A^T(Ax_k-b) - \delta\nu B^T(Bx_k-d_k+v_k)$$

- PDFP2O: (1a) is replaced by

$$x_{k+1} = x_k - \delta A^T(Ax_k-b) - \delta\nu B^T(Bx_k-d_k+v_k) - \delta^2\nu A^TAB^T(d_k-Bx_k)$$

[4] Alternating split Bregman (Goldstein-Osher,08') (ADMM)
[5] Split Inexact Uzawa, Zhang-Burger-Osher, 10'

Example: Image restoration with total variation

- Image superresolution: subsampling operator A is implemented by taking the average of every $d \times d$ pixels and sampling the average, if a zoom-in ratio d is desired.
- CT reconstruction: A is parallel beam radon transform.
- Parallel MRI:
  - Observation model:
    $$b_j = DFS_j x + \mathbf{n}$$
    where b_j is the vector of measured Fourier coefficients at receiver j, D is a diagonal downsampling operator, F is the Fourier transform, S_j corresponds to diagonal coil sensitivity mapping for receiver j and $\mathbf{n}$ is gaussian noise.
  - Let $A_j = DFS_j$, we can recover images x by solving

$$x^* = \arg\min_{x \in \mathbb{R}^n} \quad \mu \|Bx\|_1 + \frac{1}{2} \sum_{j=1}^{N} \|A_j x - b_j\|_2^2$$

Super-resolution Results

Figure : Super resolution results from 128×128 image to 512×512 image by ASB, CP, SIU and PDFP^{2}O corresponding to tolerance error $\varepsilon = 10^{-4}$

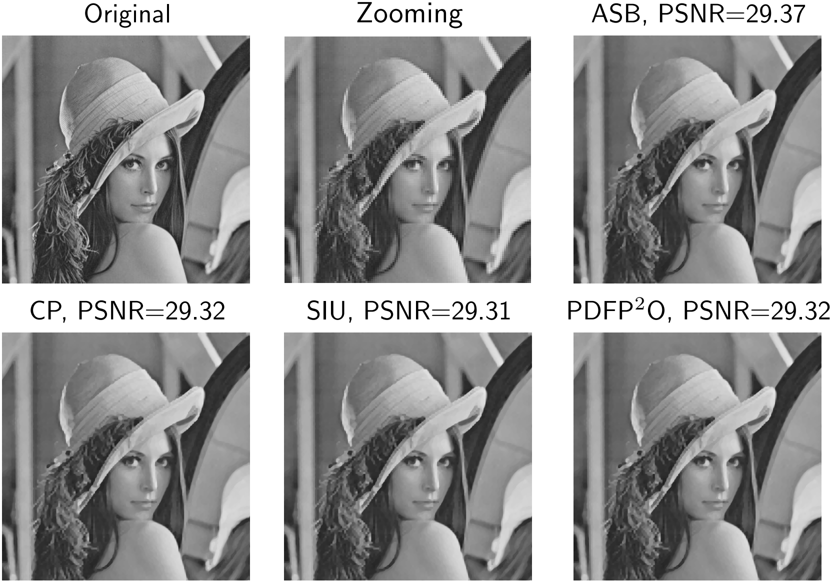


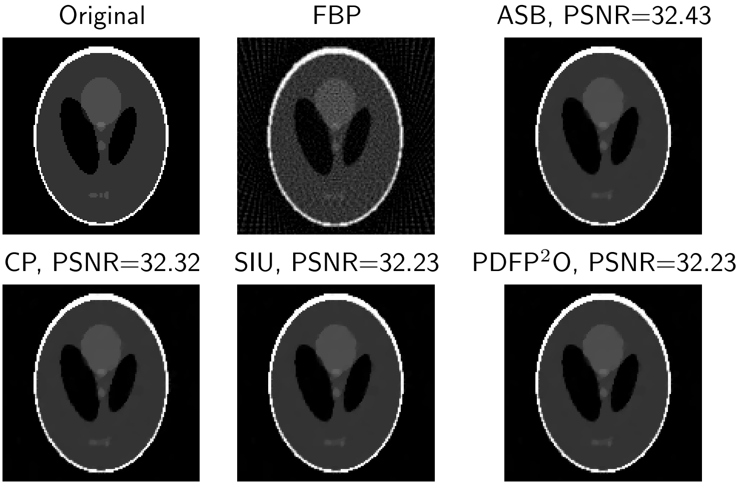
Image superresolution

Table : Performance comparison among ASB, CP, SIU and PDFP^{2}O for image superresolution. For ASB, $N_I = 2$, $\nu = 0.01$. For CP, $N_I = 1$, $\sigma = 0.005$. For SIU, $\delta = 24$, $\nu = 0.006$. For PDFP^{2}O, $\gamma = 30$, $\lambda = 1/6$.

	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
ASB	(21, 1.58, **29.34**)	(74, 5.55, **29.38**)	(254, 19.14, **29.37**)
CP	(46, 1.94, 28.97)	(150, 6.35, 29.24)	(481, 20.37, 29.32)
SIU	(42, 1.14, 28.91)	(141, 3.78, 29.22)	(446, 12.45, 29.31)
PDFP^{2}O	(38, **0.97**, 28.98)	(128, **3.25**, 29.25)	(417, **10.59**, 29.32)

Computerized tomography (CT) reconstruction

Figure : A tomographic reconstruction example for a 128×128 image, with 50 projections corresponding to tolerance error $\varepsilon = 10^{-4}$



Computerized Tomography (CT) reconstruction

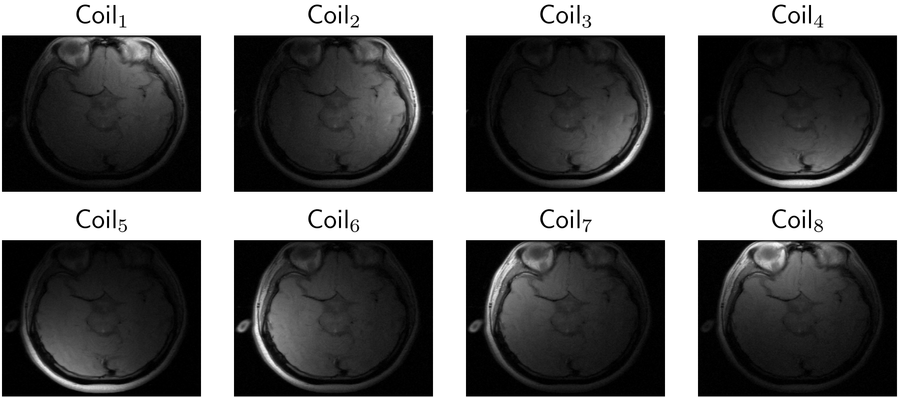
Table : Performance evaluation comparison among ASB, CP, SIU and PDFP^{2}O in CT reconstruction. For ASB, CP, SIU, empirically "best" parameter sets are used.

	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
ASB$_1$	(129, 2.72, 31.97)	(279, 5.87, 32.43)	(1068, 24.49, 32.45)
ASB$_2$	(229, 4.79, 31.75)	(345, 7.24, 32.26)	(492, 10.33, 32.41)
CP	(167, 3.37, 31.80)	(267, 5.40, 32.32)	(588, 12.73, 32.45)
SIU	(655, 4.61, 31.70)	(971, 6.84, 32.23)	(1307, 9.20, 32.38)
PDFP^{2}O	(655, 4.61, 31.70)	(971, 6.81, 32.23)	(1307, 9.15, 32.38)

Parallel MRI[6]

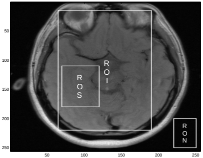
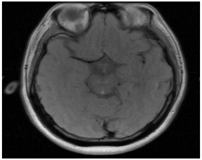
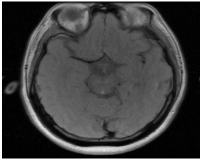
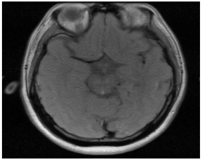
Figure : In-vivo MR images acquired eight-channel head data

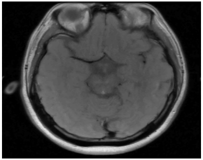
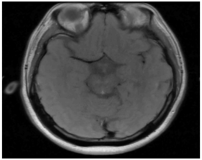
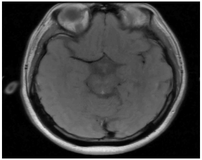
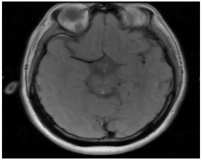
(b)



[6] Ji J X, Son J B and Rane S D 2007 PULSAR: a MATLAB toolbox for parallel magnetic resonance imaging using array coils and multiple channel receivers

Parallel MRI reconstruction

	SOS	SENSE	SPACE-RIP	GRAPPA
R=2				
AP	0	0.001939	0.001939	0.001624
SNR	36.08	31.08	31.08	29.06
time	0.19	5.94	155.88	44.85

	ASB	CP	SIU	PDFP^{2}O
R=2				
AP	0.000811	0.000823	0.000823	0.000822
SNR	39.20	39.27	39.26	39.36
time	1.81	3.72	1.87	1.84

Extensions

Extensions

- With convex constraints:

$$\underset{x\in C}{\text{arg min}} \quad (f_1 \circ B)(x) + f_2(x),$$

- Nonconvex optimization

$$\text{arg min} \quad (f_1 \circ B)(x) + f_2(x),$$

  where $f_2(x)$ is nonconvex (for example nonlinear least square). Application: nonlinear inverse problem
- Acceleration and preconditioning.

With extra convex constraints

$$\arg\min_{x\in C} \quad (f_1 \circ B)(x) + f_2(x),$$

Let Proj_C denote the projection onto the (closed) convex set C and Prox_{f_1} denote the proximal point solution.

- Convert the problem to
$$\arg\min_{x} \quad (\tilde{f}_1 \circ \tilde{B})(x) + f_2(x),$$
where $\tilde{B} = \begin{pmatrix} B \\ I \end{pmatrix}, \quad (\tilde{f}_1 \circ \tilde{B})(x) = (f_1 \circ B)(x) + \chi_C(x).$
- Apply PDFP2O, we obtain the scheme (infeasible scheme)

$$\begin{cases} v_{k+1} = (I - \mathrm{prox}_{\frac{\gamma}{\lambda} f_1})(B(x_k - \gamma \nabla f_2(x_k)) + (I - \lambda B B^T)v_k - \lambda B y_k), \\ y_{k+1} = (I - \mathrm{proj}_C)((x_k - \gamma \nabla f_2(x_k)) - \lambda B^T v_k + (1-\lambda) y_k), \\ x_{k+1} = x_k - \gamma \nabla f_2(x_k) - \lambda B^T v_{k+1} - \lambda y_{k+1}. \end{cases}$$

- Convergence is derived directly from PDFP2O.

Feasible iterative scheme

$$\begin{cases} y_{k+1} = \mathsf{Proj}_C(x_k - \gamma \nabla f_2(x_k) - \lambda B^T v_k) \\ v_{k+1} = (I - \mathsf{Prox}_{\frac{\gamma}{\lambda} f_1})(B y^{k+1} + v_k) \\ x_{k+1} = \mathsf{Proj}_C(x_k - \gamma \nabla f_2(x_k) - \lambda B^T v_{k+1}) \end{cases}$$

Equivalent iteration:

$$\begin{cases} y_{k+1} = \mathsf{Proj}_C(x_k - \gamma \nabla L(x_k, \bar{v}_k)) \\ \bar{v}_{k+1} = \arg\max L(y_{k+1}, v) - \frac{\gamma}{2\lambda} \|v - \bar{v}_k\|^2 \\ x_{k+1} = \mathsf{Proj}_C(x_k - \gamma \nabla L(x_k, \bar{v}_{k+1})) \end{cases}$$

where $L(x, v) = f_2(x) - f_1^*(v) + \langle Bx, v \rangle, \bar{v}_k = \frac{\lambda}{\gamma}(v_k)$

Convergence for feasible iterative scheme

For $x \in C, v \in S$, define

$$T(x,v): \begin{cases} T_1(x,v) = \mathrm{Prox}_{\frac{\lambda}{\gamma} f_1^*}(\frac{\lambda}{\gamma} B \circ \mathrm{Proj}_C(x - \gamma \nabla f_2(x) - \gamma B^T v) + v) \\ T_2(x,v) = \mathrm{Proj}_C(x - \gamma \nabla f_2(x) - \gamma B^T \circ T_1(x,v)) \end{cases}$$

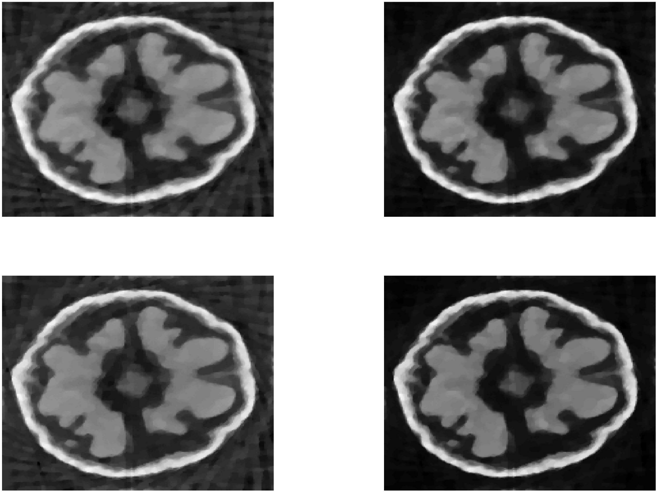
- The solution pair (x^*, v^*) satisfy $T(x^*, v^*) = (v^*, x^*)$.
- If $0 < \gamma < 2\beta$, $0 < \lambda < \frac{1}{\|BB^T\|}$, then the sequence converges.

Related reference:

Krol A Li S Shen L and Xu Y 2012 Preconditioned Alternating Projection Algorithms for Maximum a Posteriori ECT Reconstruction Inverse Problem 28 115005.

Example: CT reconstruction with real data

PDFP2O PDFP2OC



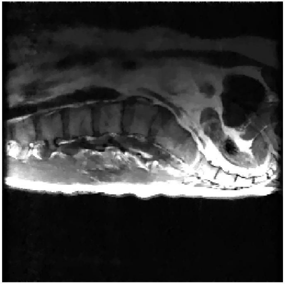
50 steps

100 steps

Figure : Real CT reconstruction with nonnegative constraint (40 projections)

Example: Parallel MRI with nonnegative constraint

SNR=38.06, 20.1s



SNR=39.55, 10.3s

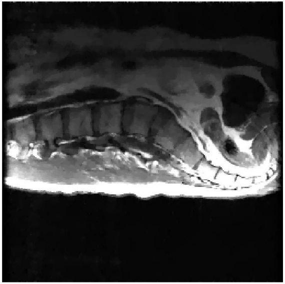


Figure : Subsampling $R = 4$

Nonconvex optimization for nonlinear inverse problem

$$\min_{x} \|F(x) - y\|_2^2 + \lambda \|Wx\|_1, \quad \text{subject} \ \ \text{to} \ \ l \leq x \leq u.$$

Let $f(x) = \|F(x) - y\|_2^2$.

$$\begin{cases} y^k = \text{Prox}_{\gamma_C}(x^k - \tau\big(\nabla_x f(x^k) + \lambda\mu W^T b^k\big)), \\ h^k = b^k + Wy^k, \\ b^{k+1} = h^k - shrink(h^k, 1/\mu), \\ x^{k+1} = \text{Prox}_{\gamma_C}(x^k - \tau\big(\nabla_x f(x^k) + \lambda\mu W^T b^{k+1}\big)), \end{cases}$$

Application for Quantitative photo-acoustic reconstruction[8]

Reconstruction time: 180 s *v.s* 887s SplitBregman (L-BFGS for the subproblem) [7]

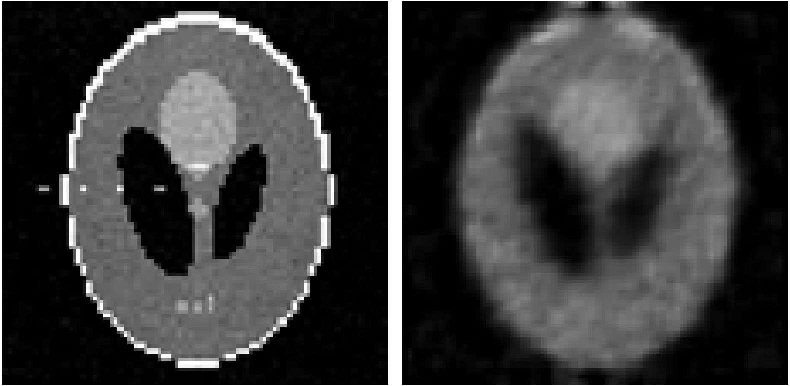


Figure : Quantitative photo-acoustic reconstruction for the absorption and (reduced) scattering.

[7] H. Zhao, S. Osher and H. Zhao: Quantitative Photoacoustic Tomography, 2012

[8] Ongoing joint work with X. Zhang, W Zhou and H. Gao

Summary and perspectives

- First order methods are efficient and enjoy comparative numerical performance, especially when the parameters are properly chosen.
- For both ASB and CP, inexact solver such as CG method can be applied in practice. However the choice of inner iteration is rather ad-hoc. For both SIU and PDFP^{2}O, the iteration schemes are straightforward since only one inner iteration is used and easy to implement in practice and can be easily, especially when one extra constraint is present. It can be also extended to low rank matrix or other more complex sparsity reconstruction.
- Parameter choices for all the methods. The rules of parameter selection in PDFP^{2}O has some advantages.
- The convergence and the relation of convergence rate and condition number of the operators is clearly stated under this theoretical framework.
- Future work: convergence for nonconvex problems; acceleration and preconditioning; parellel and distributed, decentralized computing .

- Reference P. Chen, J. Huang and X. Zhang, A Primal-Dual Fixed Point Algorithm for Convex Separable Minimization with Applications to Image Restoration. Inverse problems, 29 (2), 2013

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Thank you for your attention !