Inverse Problems in Interferometric Phase Imaging

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Project FCT: UID/EEA/50008/2013

Mathematics and Image Analysis (MIA), Paris, January, 2016

Phase estimation from interferometric measurements

Problem: given a set of observations $e^{j\phi_p} \equiv (\cos\phi_p, \sin\phi_p)$, determine ϕ_p (up to a constant) for $p \in \mathcal{V} \equiv \{1, \ldots, n\}$

$e^{j\phi_p}$ is 2π- periodic → nonlinear and ill-posed inverse problem

Continuous/discrete flavor: $\phi = \mathcal{W}(\phi) + 2k\pi \qquad \mathcal{W} : \mathbb{R} \to [\pi, \pi[$

Phase Unwrapping (PU)

↓

Estimation of $k \in \mathbb{Z}$

Phase Denoising (PD)

↓

Estimation of $\mathcal{W}(\phi) \in [\pi, \pi[$

(wrapped phase)

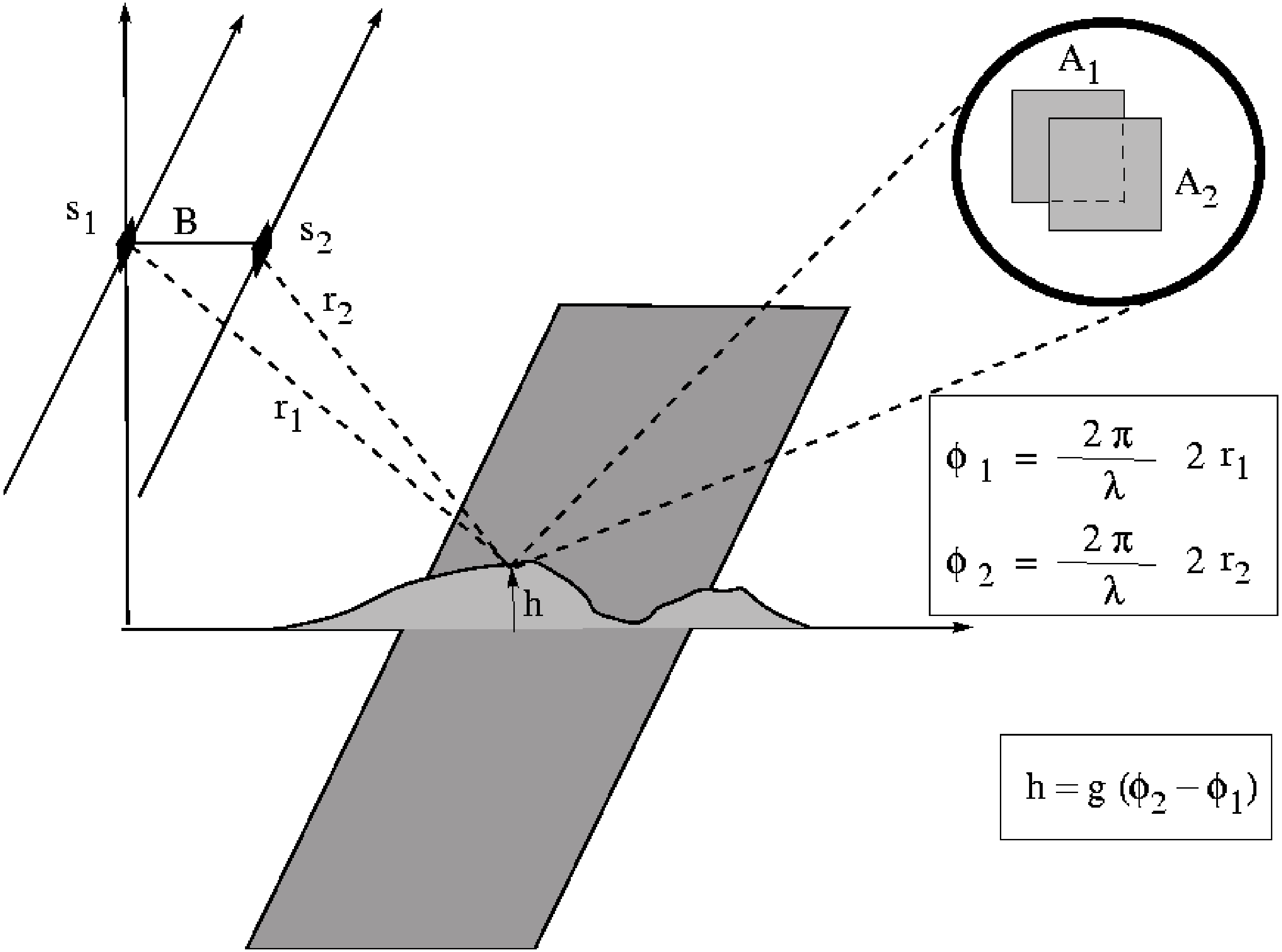
Outline

- Interferometric phase imaging. Examples
- Absolute phase estimation
- Phase unwrapping
- Interferometric phase denoising via sparse regression
- Multisource phase estimation
- Concluding remarks

Applications

- Synthetic aperture radar/sonar
- Magnetic resonance imaging
- 3D surface imaging from structured light
- High dynamic range photography
- Diffraction tomography
- Optical interferometry
- Tomographic phase microscopy
- Doppler echocardiography
- Doppler weather radar

Absolute phase estimation in InSAR (Interferometric SAR)



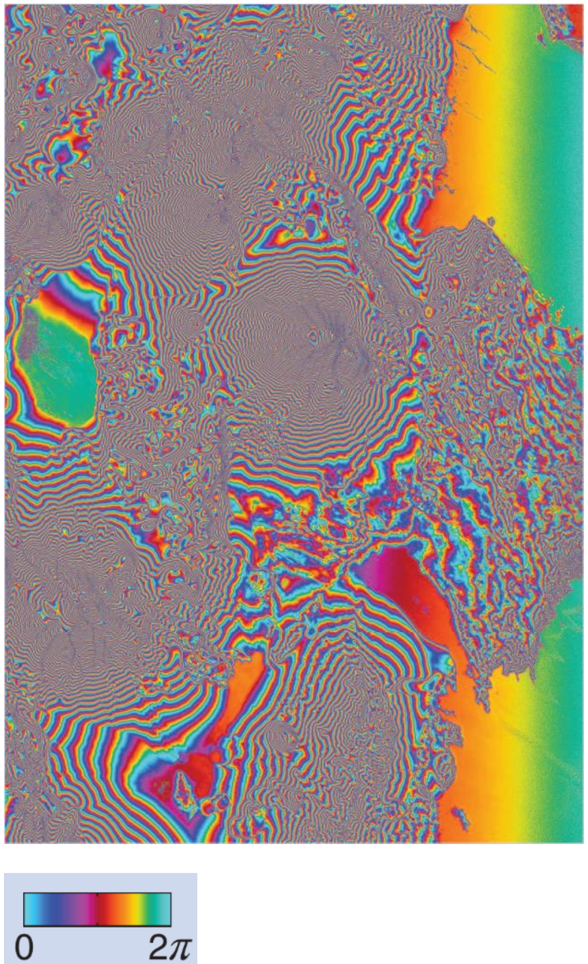
InSAR Problem: Estimate $\phi_2 - \phi_1$ from signals read by s_1 and s_2

InSAR Example

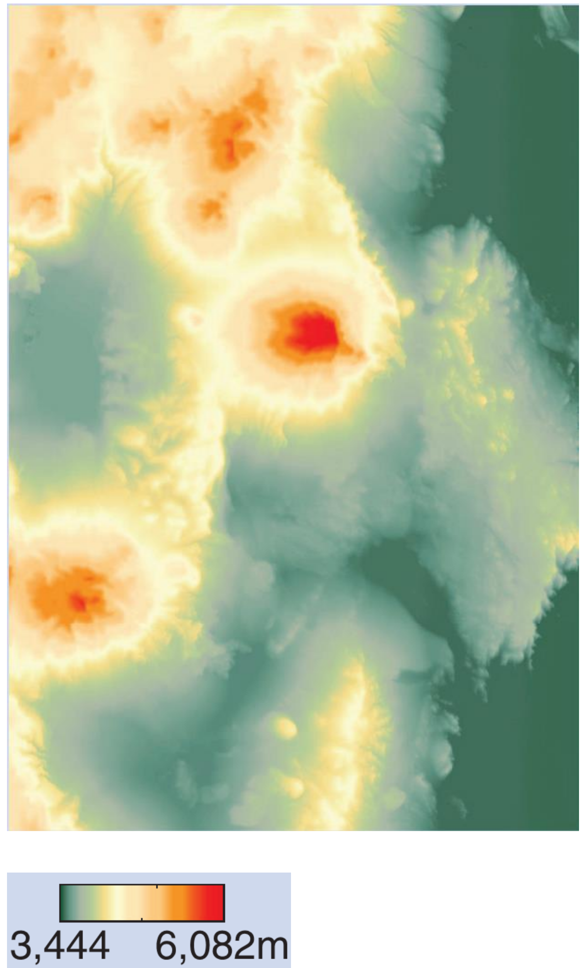
Atacama desert (Chile)

(from [Moreira et *al.*,13])

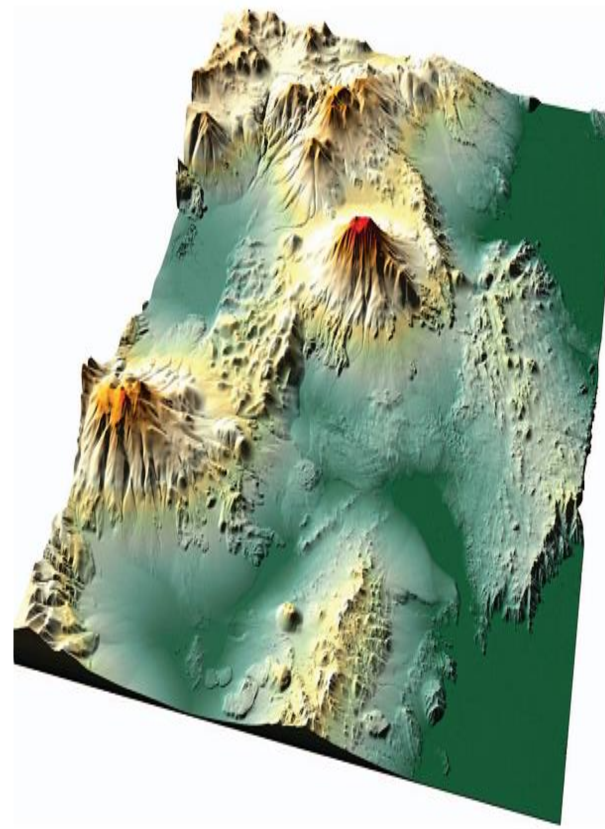
Interferogram $\mathcal{W}(\phi_1 - \phi_2)$



Unwrapped phase $\phi_1 - \phi_2$

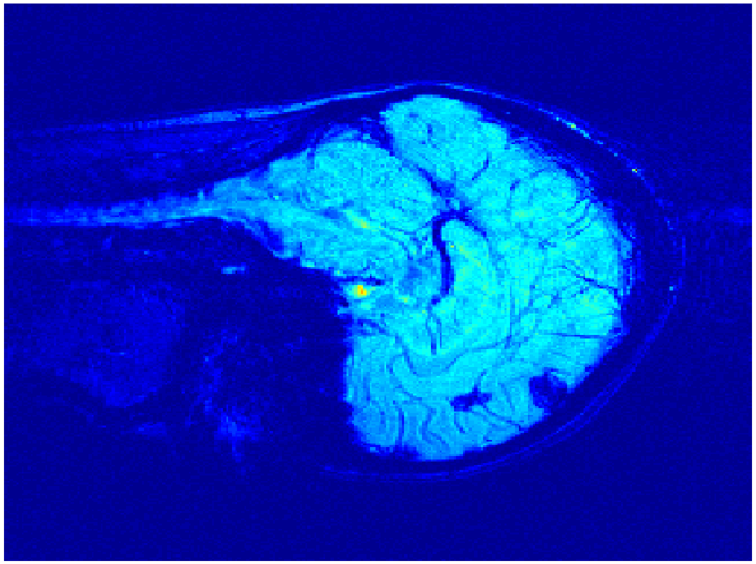


Geocoded digital elevation model (DEM)

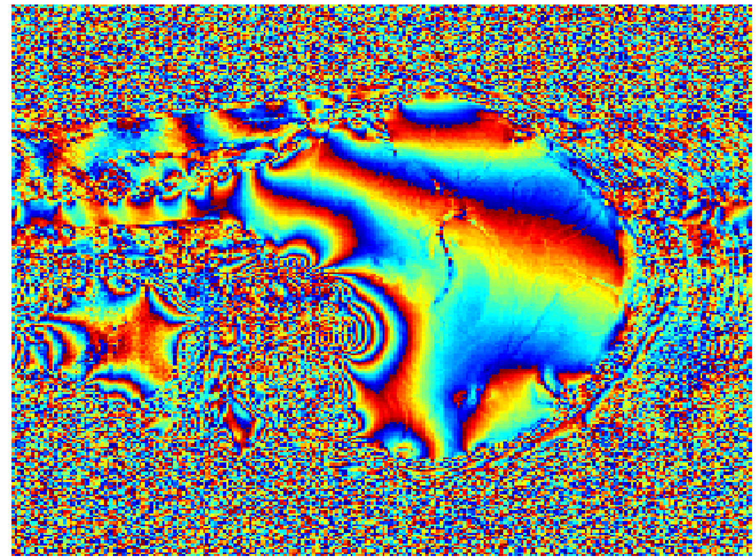


Magnetic resonance imaging (MRI)

Intensity



Interferometric phase



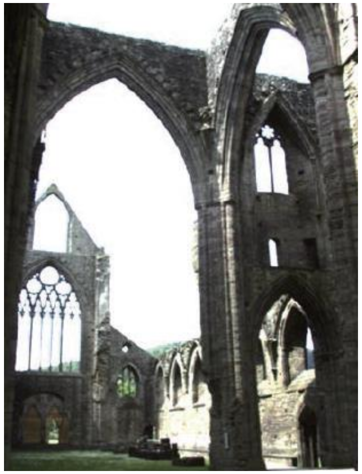
Interferomeric phase

- measure temperature
- visualize veins in tissues
- water-fat separation
- map the principal magnetic field

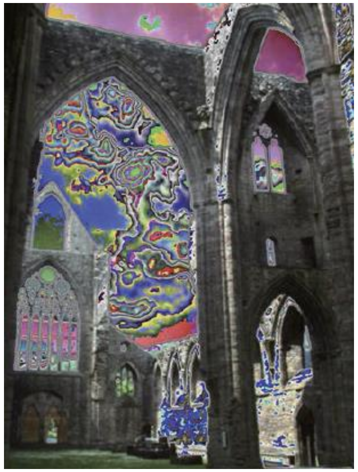
High dynamic range photography

(from [Zhao et *al.*, 15])

Intensity camera



Modulo camera



$$I_N = \mod\left(I, 2^N\right)$$

Unwrapped image (tone-mapped)

3D surface imaging from structured light

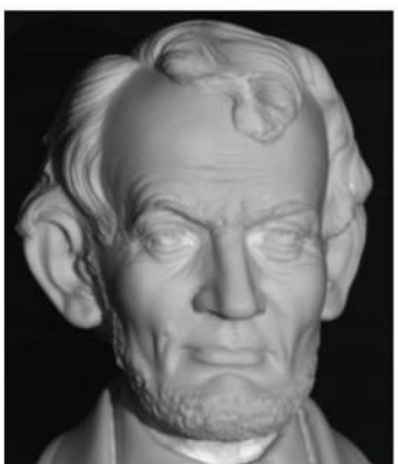
(from [Huang et *al.*, 06])

Fringe images

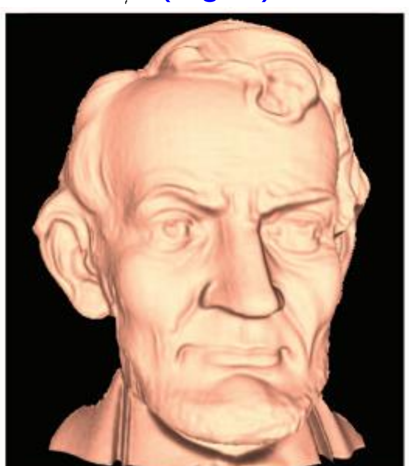
$\phi_{r_1} = -120^\circ$ $\phi_{r_2} = 0^\circ$ $\phi_{r_3} = 0^\circ$

$$I_k = b_0 + b_1 \cos(\phi - \phi_{r_k})$$

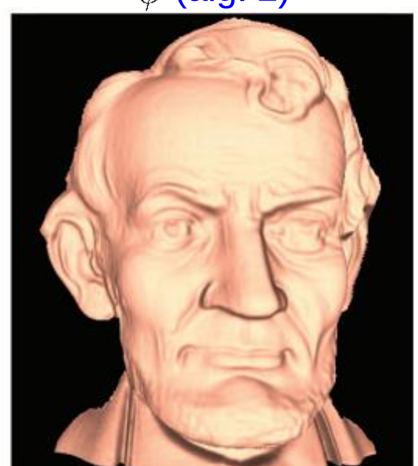
Original



$\widehat{\phi}$ (alg. 1)



$\widehat{\phi}$ (alg. 2)

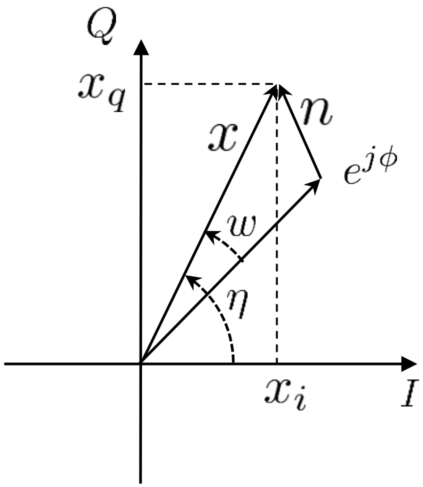


Forward problem: sensor model

$$x_i = \cos\phi + n_i \qquad n = (n_i, n_q)$$

$$x_q = \sin\phi + n_q \qquad x = (x_i, x_q)$$

$$n_i, n_q \sim \mathcal{N}(0, \sigma^2)$$ independent



$$\eta = \mathcal{W}(\phi) + w$$

$$\mathcal{W}(\phi), w \in [-\pi, \pi[$$

Data likelihood

$$p(x|\phi) \propto c\, e^{\lambda \cos(\phi - \eta)}$$

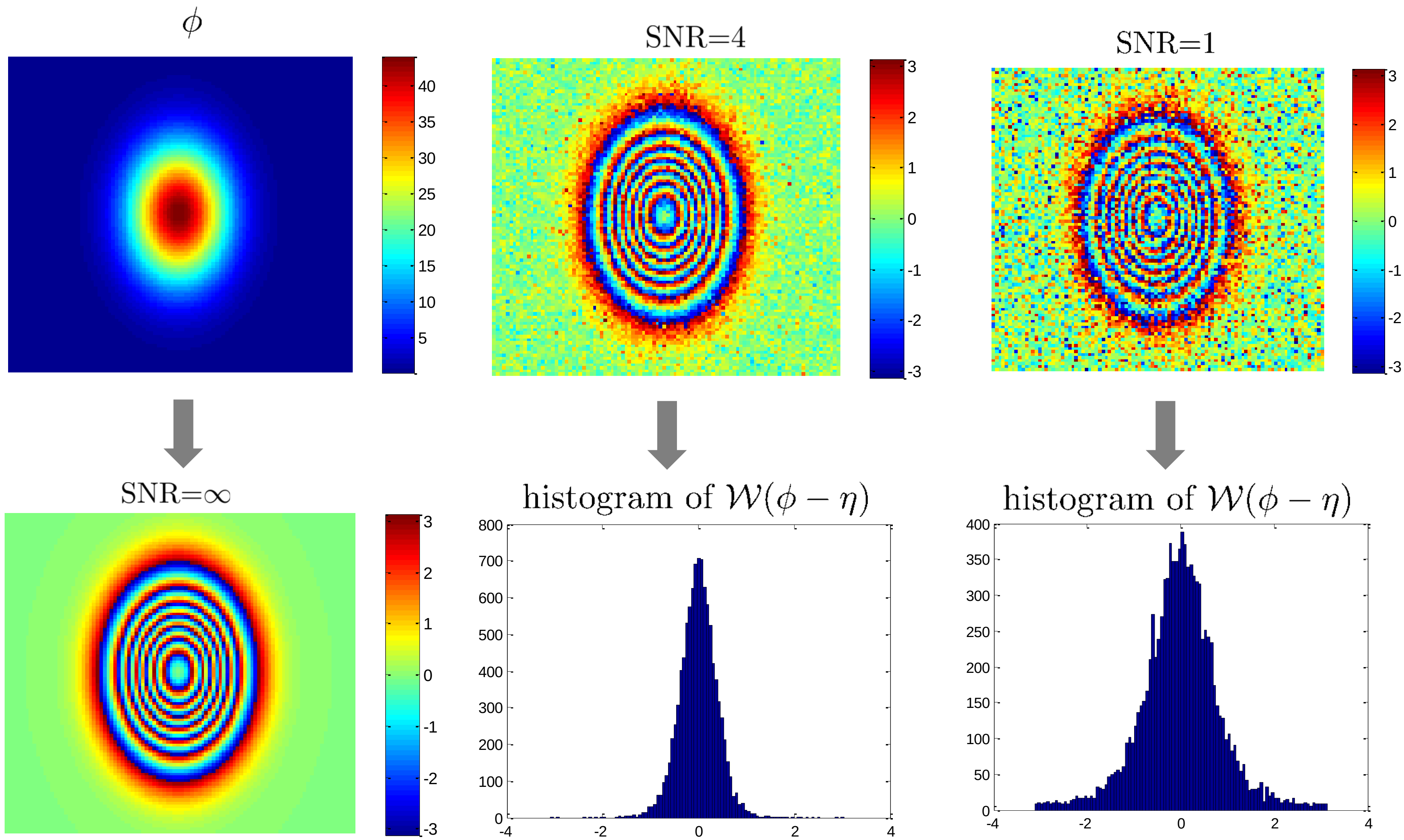
$$\eta = \arg(x) \quad \lambda = \frac{2|x|}{\sigma^2}$$

$$\Rightarrow \quad \widehat{\phi}_{ML} = \eta + 2k\pi$$

Simulated Interferograms

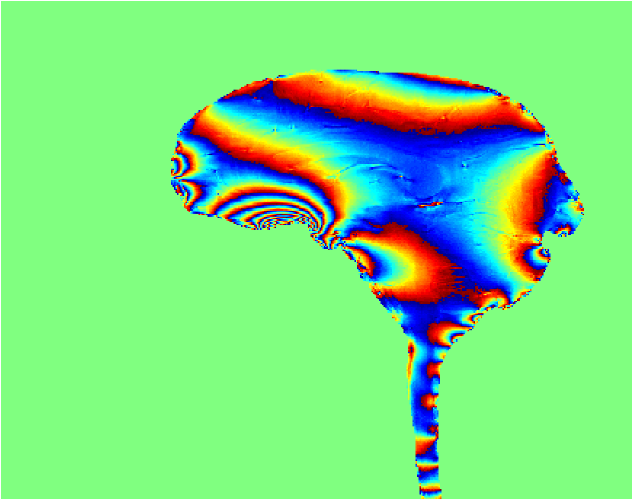
Images: $\eta = \arg(e^{j\phi} + n)$

$$\mathrm{SNR} = \frac{1}{2\sigma^2}$$

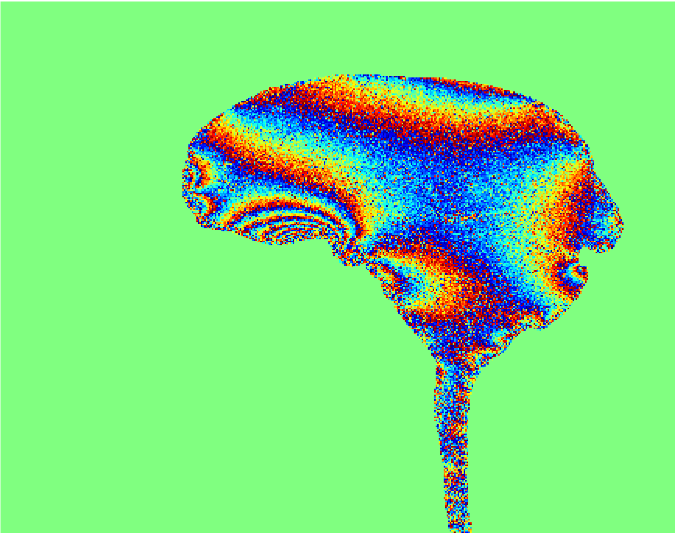


Real interferograms

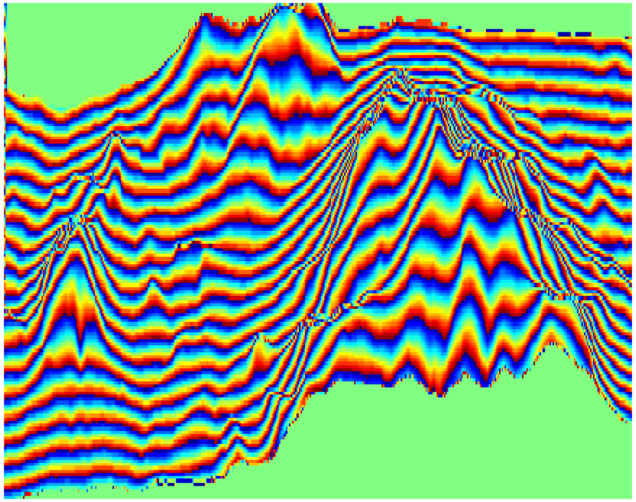
MRI



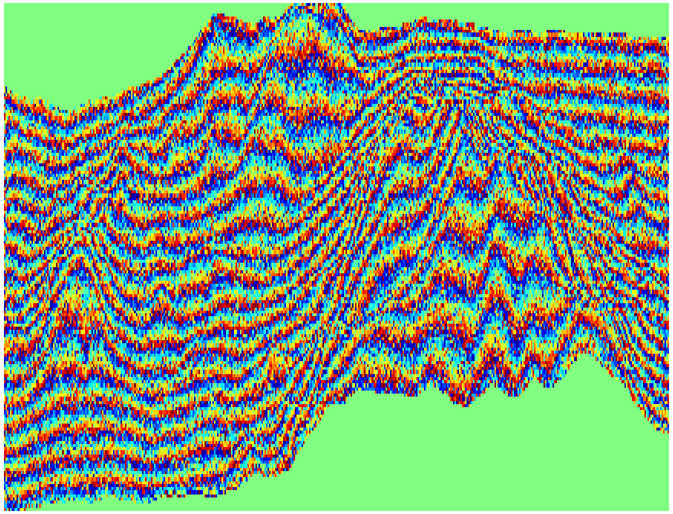
MRI



InSAR



InSAR



Bayesian absolute phase estimation

Data term: $p(\mathbf{x}|\boldsymbol{\phi}) = \prod_{p\in\mathcal{V}} p(x_p|\phi_p)$ Prior term: $p(\boldsymbol{\phi}) = \frac{1}{Z} e^{-U(\boldsymbol{\phi})}$

Ex: pairwise interactions $U(\boldsymbol{\phi}) = \sum_{\{p,q\}\in\mathcal{E}} U_{pq}(\phi_p - \phi_q)$

- $\mathcal{E} = \{\{p,q\} \,:\, p \sim q\}$ clique set
- U_{pq} clique potential

U_{pq} convex	U_{pq} non-convex
⬇	⬇
Enforces smoothness	Enforces piecewise smoothness (discontinuity preserving)

Estimation criteria

Maximum a posteriori (MAP) $\widehat{\phi} \in \arg\max_{\boldsymbol{\phi}\in\mathbb{R}^n} \; p(\mathbf{x}|\boldsymbol{\phi})p(\boldsymbol{\phi}) = \arg\min_{\boldsymbol{\phi}\in\mathbb{R}^n} E(\boldsymbol{\phi})$

$$E(\boldsymbol{\phi}) = \sum_{p\in\mathcal{V}} -\lambda_p \cos(\phi_p - \eta_p) + U(\boldsymbol{\phi})$$

E is hard to optimize due to the sinusoidal data terms

Popular approaches to absolute phase estimation

- Reformulation as linear observations in non-Gaussian noise
- Interferometric phase denoising + phase unwrapping

Phase differences

Wrapped difference of wrapped phases: $\eta = \mathcal{W}(\phi) + w$

$$\mathcal{W}(\eta_p - \eta_q) = (\phi_q - \phi_q) + \boxed{(w_p - w_q)} + \boxed{2\pi l_{p,q}}$$

additive noise distributed in $[-2\pi, 2\pi[$

wrap errors due to discontinuities, high phase rate, and noise

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- In the absence of noise, $l_{p,q} = 0$ if $|\phi_q - \phi_q| < \pi$ (Itoh condition)
- In most applications $P(|\phi_q - \phi_q| \geq \pi)$ is small but positive
- $l_{p,q} = 0 \text{ for } \{p,q\} \in \mathcal{E}$ if $\max_{\{p,q\}\in\mathcal{E}} |\phi_p - \phi_q| + \max_{\{p,q\}\in\mathcal{E}} |w_p - w_q| < \pi$
- Number of wrap errors increases with σ. If $w_p \sim \mathcal{N}(0, \sigma^2)$, then

$$\mathbb{E}[\max_{\{p,q\}\in\varepsilon} |w_p - w_q|]) \geq \mathbb{E}[\max_{\{p,q\}\in\varepsilon} (w_p - w_q)] = O\left(\sigma\sqrt{\log|\mathcal{E}|}\right)$$

Absolute phase estimation: linear observations in non-Gaussian noise

$$\mathbf{y} = \mathcal{W}(\mathbf{D}\boldsymbol{\eta}) \qquad \mathbf{D}: \mathbb{R}^n \rightarrow \mathbb{R}^{2n} \text{ – discrete gradient}$$

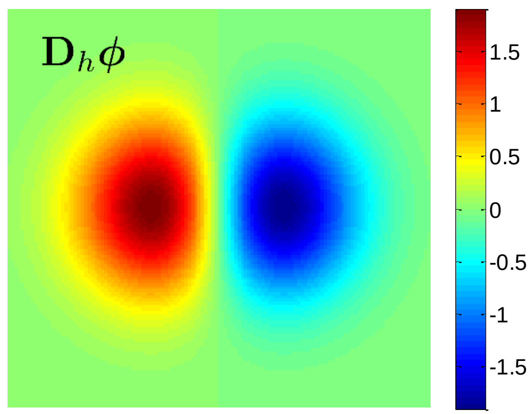
$\mathbf{w}_\eta$ – interferometric noise

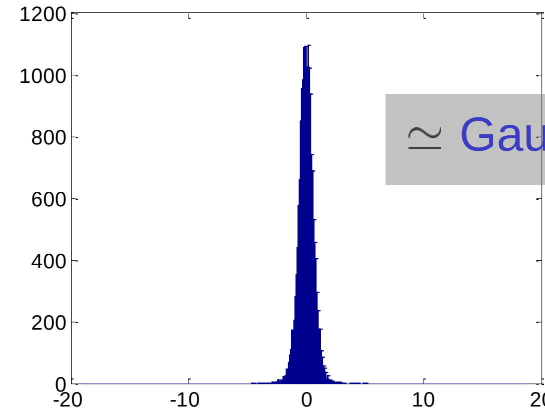
$\mathbf{w}_\pi$ – wrap errors

$$\mathbf{y} = \mathbf{D}\boldsymbol{\phi} + \mathbf{w}_\eta + \mathbf{w}_\pi$$

Histograms of $\mathbf{y} - \mathbf{D}\boldsymbol{\phi} = \mathbf{w}_\eta + \mathbf{w}_\pi$ for a Gaussian phase surface

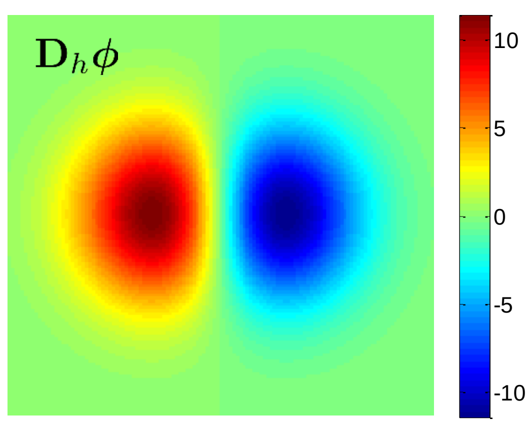
$|\phi_q - \phi_q| < \pi$

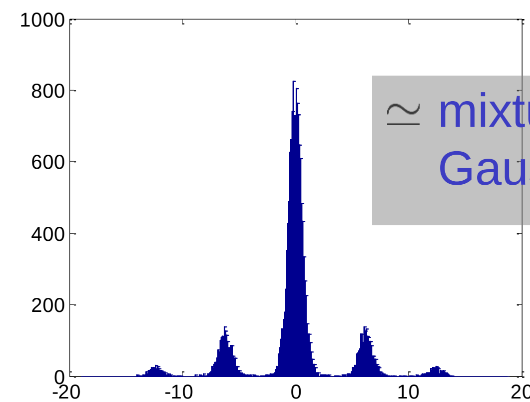




$\simeq$ Gaussian

$|\phi_q - \phi_q| \geq \pi$





$\simeq$ mixture of Gaussians

Formulation based on the linear observation model (LOM)

Minimum ℓ_p norm $0 < p < 2$ [Ghiglia & Pritt, 98]

Algorithms

$$\min_{\phi \in \mathbb{R}^n} \|\mathbf{y} - \mathbf{D}\phi\|_{p,Q} \quad \text{s.t.} \quad \mathbf{A}\phi = \mathbf{b}$$

IRLS, MM
[Lange & Fessler., 95]

Regularized ℓ_1 norm (convex) [Gonzalez & Jacques, 15]

$$\min_{\phi, \mathbf{u} \in \mathbb{R}^n} \|\mathbf{W}\phi\|_1 \text{ s.t.} \begin{cases} \|\mathbf{y} - \mathbf{D}(\phi + \mathbf{u})\|_1 \leq \varepsilon_\pi \\ \|\mathbf{u}\|_2 \leq \varepsilon_w \\ \mathbf{A}\phi = \mathbf{b} \end{cases}$$

PD
[Chambolle, Pock, 11]

Adaptive regularized ℓ_2 norm [Kamilov et al.,15]

$$\min_{\phi \in \mathbb{R}^n} \sum_{i=1}^{n} q_i^t \|\mathbf{y}_i - \mathbf{D}_i\phi\|_2 + \tau \|\mathbf{H}_i\phi\|_* \quad \text{s.t.} \quad \mathbf{A}\phi = \mathbf{b}$$

Seq. of ADMM subproblems (q_i^t)

SALSA
[Afonso, B-D, Fig., 11]

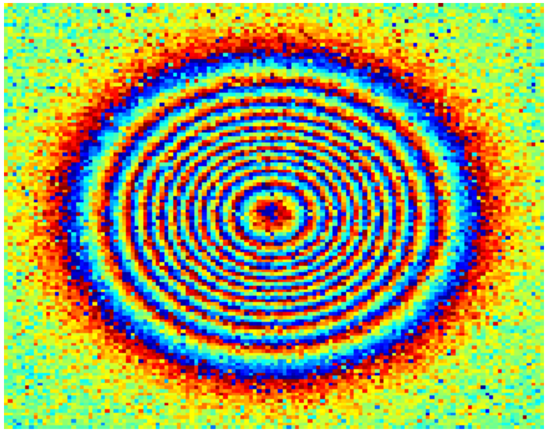
$\mathbf{D}_i : \mathbb{R}^n \to \mathbb{R}^2$ – discrete gradient

$\mathbf{y}_i = (y_{h,i}, y_{v,i})$

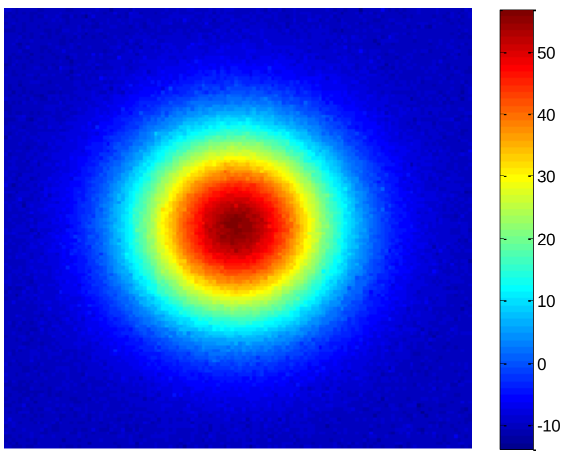
Nuclear norm
$\mathbf{H}_i : \mathbb{R}^n \to \mathbb{R}^4$ – discrete Hessian

Example: IRTV ([Kamilov et al.,15]) (SALSA implementation) $n = 128 \times 128$

$\max \phi_p = 20\pi \quad \sigma = 0.5$



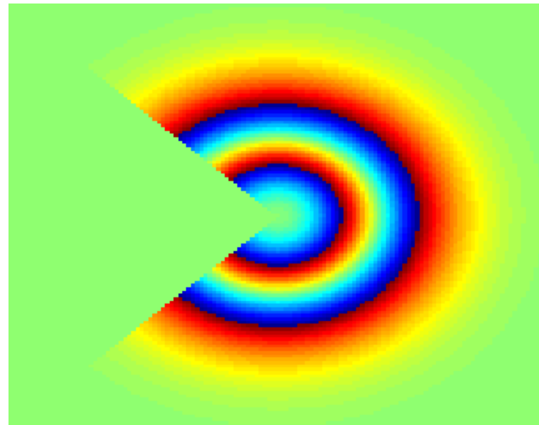
$\tau = 10^{-3}$



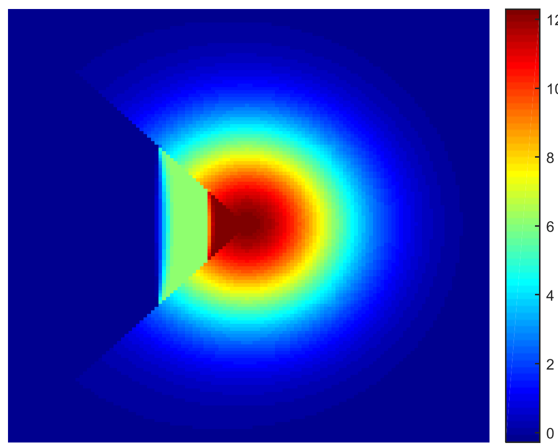
$$\text{ISNR} = \frac{2n\sigma^2}{\|\widehat{\phi} - \phi\|_F^2}$$

$$\text{ISNR} = (1.4, 1.5, -16.4)\,\text{dB}$$
$$\tau = (10^{-4}, 10^{-2}, 10^{0})$$

$\max \phi_p = 4\pi$



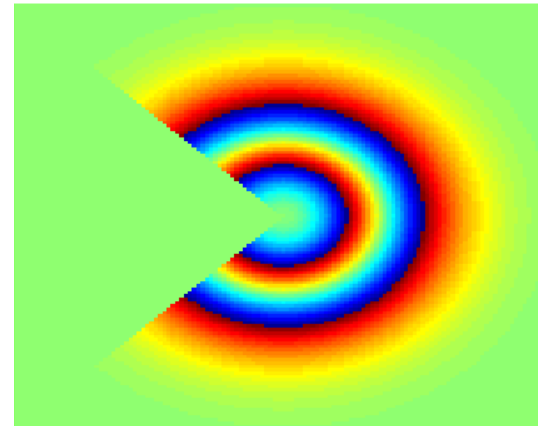
$\tau = 10^{-3}$



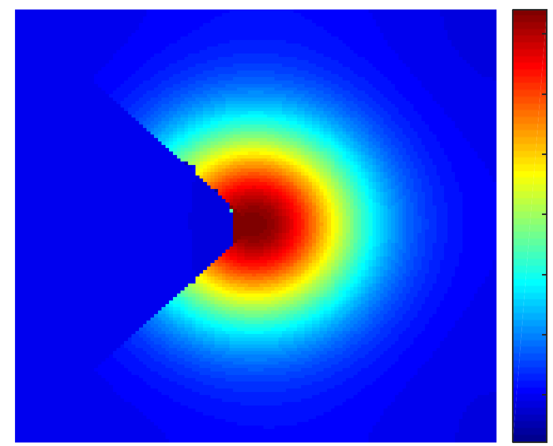
1 iter (fixed weights)

time $= 20\,\text{s}$

$\max \phi_p = 4\pi$



$\tau = 10^{-3}$



10 iters (adaptive weights)

time $= 200\,\text{s}$

A few comments on the LOM-based phase estimation

Observation model: $\mathbf{y} = \mathbf{D}\boldsymbol{\phi} + \mathbf{w}_\eta + \mathbf{w}_\pi$ $\qquad \mathbf{w}_\eta = \mathbf{D}\mathbf{w}$

- $\mathbf{D}\boldsymbol{\phi}$: self-similar
- $\mathbf{w}_\eta$: highpass
- $\mathbf{w}_\pi$: sparse and ϕ-dependent

- Regularization is challenging. Ex: Tickhonov regularization using $\|\mathbf{D}\boldsymbol{\phi}\|^2$

$$\widehat{\boldsymbol{\phi}} = \frac{1}{1+\tau}\left(\boldsymbol{\phi} + \mathbf{w} + \mathbf{D}^\dagger \mathbf{w}_\pi\right)$$

- $\mathbf{w}$: original interferometric
- $\mathbf{D}^\dagger \mathbf{w}_\pi$: wrap errors are amplified

- The wrap errors $\mathbf{w}_\pi$ due to phase discontinuities tend to be sparse and thus well modeled by ℓ_p norms with $p \leq 1$

- ℓ_1 norm (and ℓ_1 on the gradient) yields convex programs but has limited power to cope with wrap errors

→

1) Denoise (filter out $\mathbf{w}$)

2) (Use ℓ_p with $p < 1$) or ($p \geq 1$ and detect the discontinuities)

Interferometric phase denoising + phase unwrapping

Back to MAP estimate

$$\widehat{\phi} \in \arg\min_{\boldsymbol{\phi}\in\mathbb{R}^n} E(\boldsymbol{\phi}) \qquad E(\boldsymbol{\phi}) = \sum_{p\in\mathcal{V}} -\lambda_p \cos(\phi_p - \eta_p) + U(\boldsymbol{\phi})$$

Assume that: $\boldsymbol{\phi} = \{\phi_p | \phi_p = \eta_p + 2k_p\pi, p \in \mathcal{V}, k_p \in \mathbb{Z}\} \qquad (\Leftrightarrow \lambda_p \to \infty)$

Then: $$\widehat{\mathbf{k}} \in \arg\min_{\mathbf{k}\in\mathbb{Z}^n} E(\boldsymbol{\eta}, \mathbf{k}) = \arg\min_{\mathbf{k}\in\mathbb{Z}^n} U(\boldsymbol{\eta}, \mathbf{k})$$

Integer optimization

Pairwise interactions: $$U(\boldsymbol{\eta}, \mathbf{k}) = \sum_{\{p,q\}\in\mathcal{E}} V_{pq}(k_p - k_q)$$

$$V_{pq}(k_p - k_q) = U_{pq}(\eta_p - \eta_q + 2\pi(k_p - k_q))$$

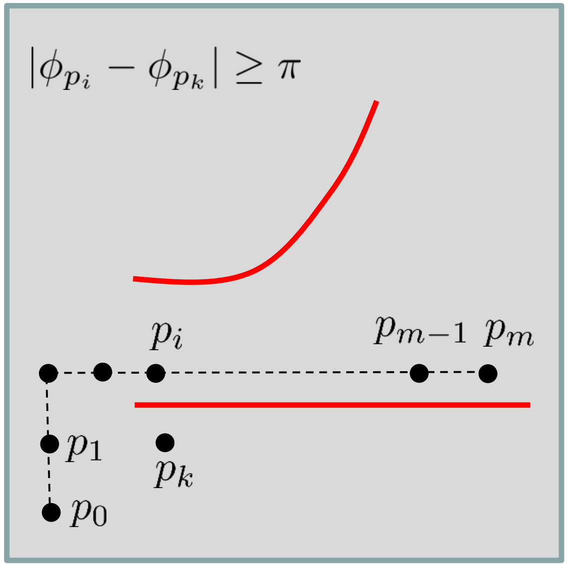
Phase unwrapping: path following methods

Assume that $|\phi_p - \phi_q| < \pi$ (Itoh condition) $\phi_p = \eta_p + 2k_p\pi$

Then $\phi_p - \phi_q = \mathcal{W}(\phi_p - \phi_q) = \mathcal{W}(\eta_p - \eta_q)$

PU $\Leftrightarrow$ summing $\mathcal{W}(\eta_p - \eta_q)$ over walks

$$\phi_{p_m} = \phi_{p_0} + \sum_{i=1}^{m} \mathcal{W}(\eta_{p_i} - \eta_{p_{i-1}})$$



Why isn't PU a trivial problem?

- Discontinuities
- High phase rate
- Noise

$|\phi_p - \phi_q| \geq \pi$

Phase unwrapping algorithms

$$E(\mathbf{k}) = \sum_{\{p,q\}\in\mathcal{E}} V_{pq}(k_p - k_q)$$

- $V_{pq}(\cdot) = |\cdot|_{2\pi-\text{quantized}}$

  [Flynn, 97] (exact) sequence of positive cycles on a graph

  [Costantini, 98] (exact) min-cost flow on a graph $(|\mathcal{V}| = n, |\mathcal{E}| = 4n)$

- $V_{pq}(\cdot) = (\cdot)^2$

  [B-D & Leitao, 01] (exact) sequence of positive cycles on a graph $(|\mathcal{V}| = n, |\mathcal{E}| = 4n)$

  [Frey et *al*., 01] (approx) belief propagation on a 1st order MRF

- $V_{pq}(\cdot)$ convex

  [B-D & Valadao, 07,09,11] (exact) ∫equence of *K* min cuts ($KT(n, 6n)$)

- $V_{pq}(\cdot)$ non-convex

  [Ghiglia, 96] LPN0 (continuous relaxation)

  [B-D & G. Valadao, 07,09,11] sequence of min cuts ($KT(n, 6n)$)

PUMA (Phase Unwrapping MAx-flow) [B-D & Valadao, 07,09,11]

Algorithm 1: PUMA

$\boldsymbol{\phi} := \boldsymbol{\eta}, \quad succes == false$

while $succes == false$ **do**

  $\boldsymbol{\delta} := \arg \min_{\mathbf{x} \in \{0,1\}^{|\nu|}} E(\boldsymbol{\phi} + 2\mathbf{x}\pi)$

  if $E(\boldsymbol{\phi} + 2\mathbf{x}\pi) < E(\boldsymbol{\phi})$ **then**

    $\boldsymbol{\phi} := \boldsymbol{\phi} + 2\boldsymbol{\delta}\pi$

  else

    $succes == false$

return $\boldsymbol{\phi}$

PUMA finds a sequence of steepest descent binary images

Convex priors $E(\mathbf{k}) = \sum V_{pq}(k_p - k_q)$

- A local minimum is a global minimum
- Takes at most *K* (range of k) iterations
- E is submodular: $2V_{pq}(0) \leq V_{pq}(1) + V_{pq}(-1)$

$\Rightarrow$ each binary optimization has the complexity of a min cut $T(n, 6n)$

PUMA: convex priors

$$E(\mathbf{k}) = \sum V_{pq}(k_p - k_q)$$

- Let ϕ be a smooth surface in the Itoh sense. That is $|\phi_p - \phi_q| < \pi$ for $\{p,q\} \in \mathcal{E}$. If $U_{pq}(x)$ is convex and strictly increasing of $|x|$, then

$$\boldsymbol{\phi} = \boldsymbol{\eta} + \widehat{\mathbf{k}} + c$$

where $\widehat{\mathbf{k}}$ is the PUMA solution

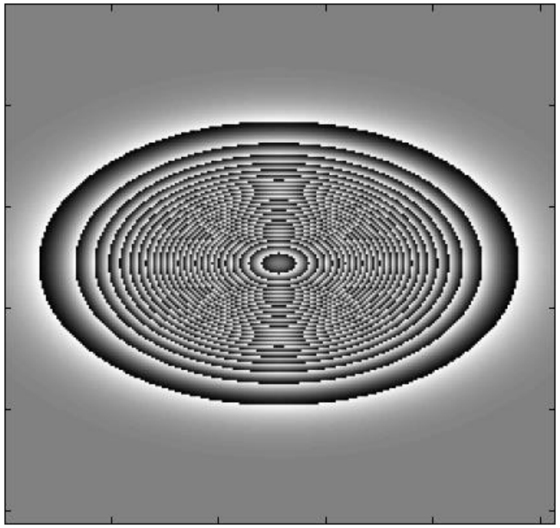
- Related algorithms

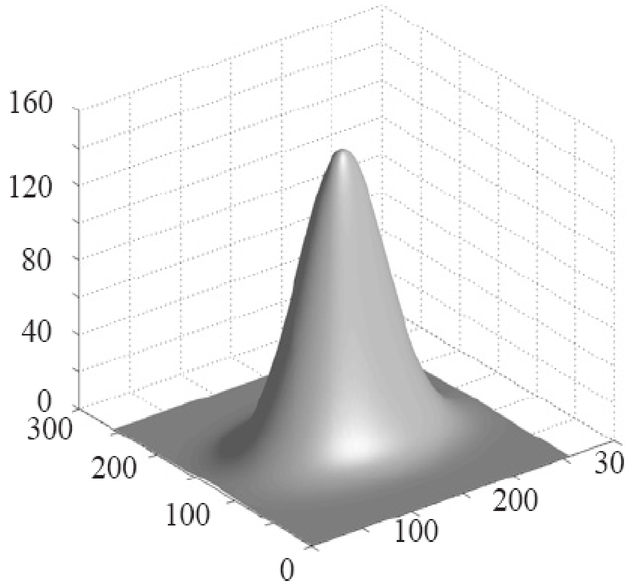
$$E(\mathbf{k}) = \sum_{p \in \mathcal{V}} D_p(k_p) + \sum_{\{p,q\} \in \mathcal{E}} V_{pq}(k_p - k_q)$$

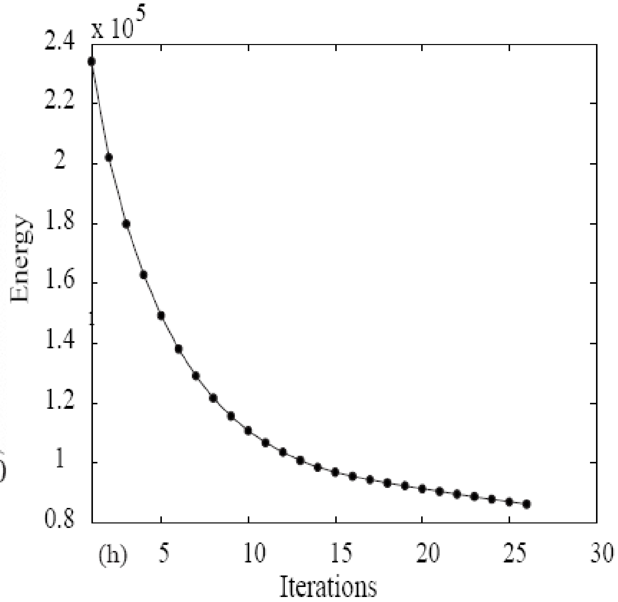
[Veksler, 99] (1-jump moves)
[Murota, 03] (steepest descent algorithm for L-convex functions)
[Ishikawa, 03] (MRFs with convex priors)
[Kolmogorov & Shioura, 05,09], [Darbon, 05] (Include unary terms)
[Ahuja, Hochbaum, Orlin, 03] (convex dual network flow problem)

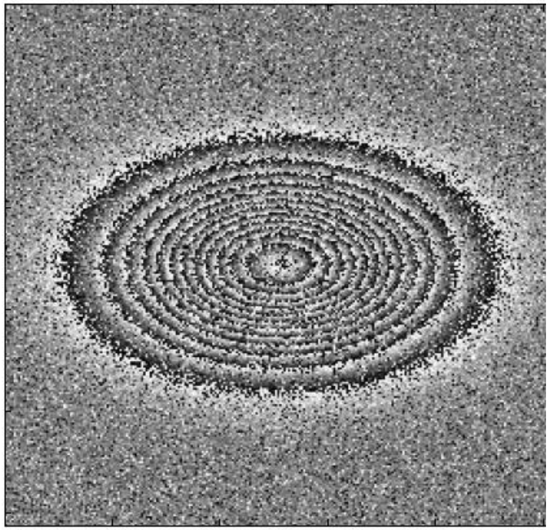
Results

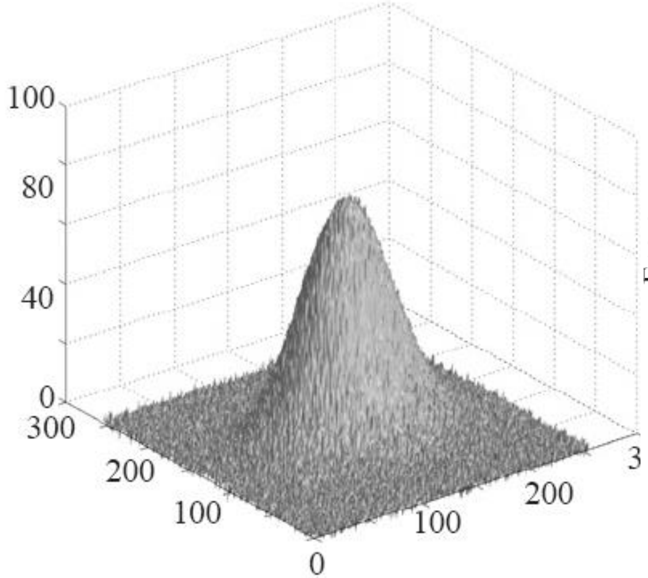
$$U_{pq}(\cdot) = (\cdot)^2$$

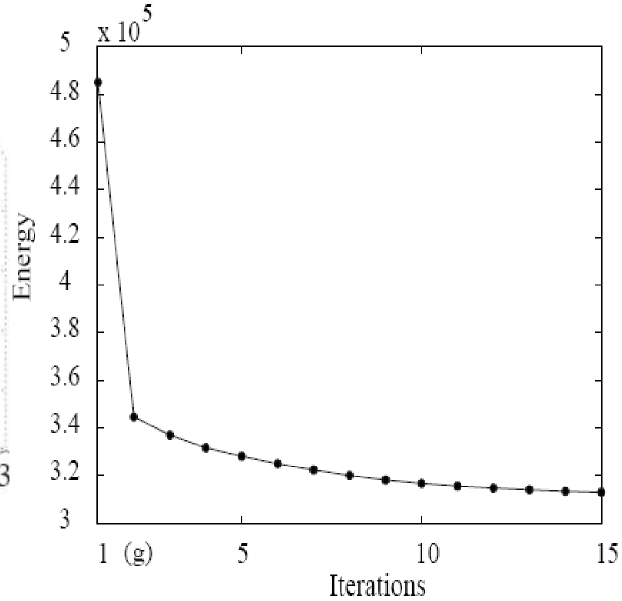






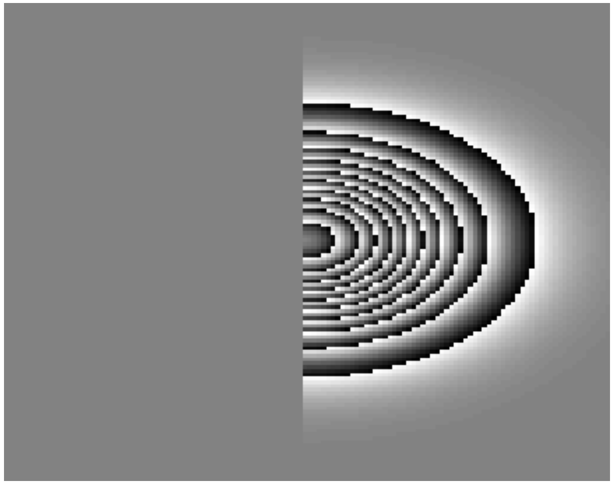




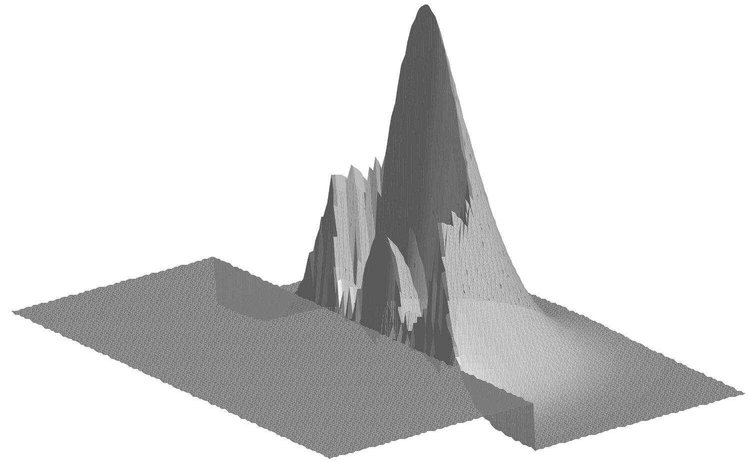


Results

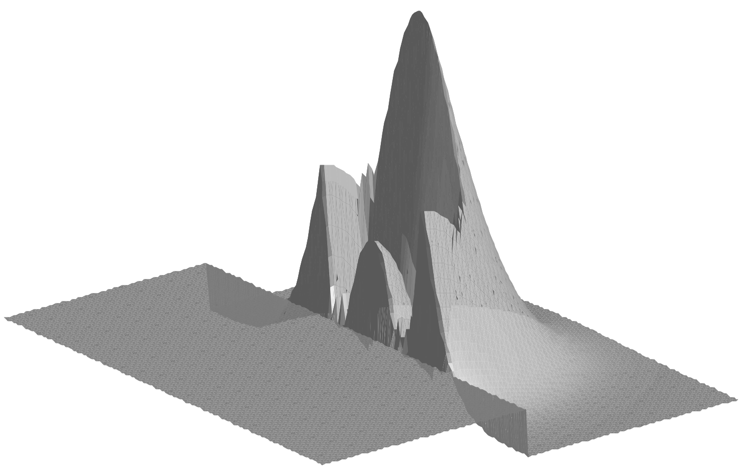
Convex priors do not preserve discontinuities



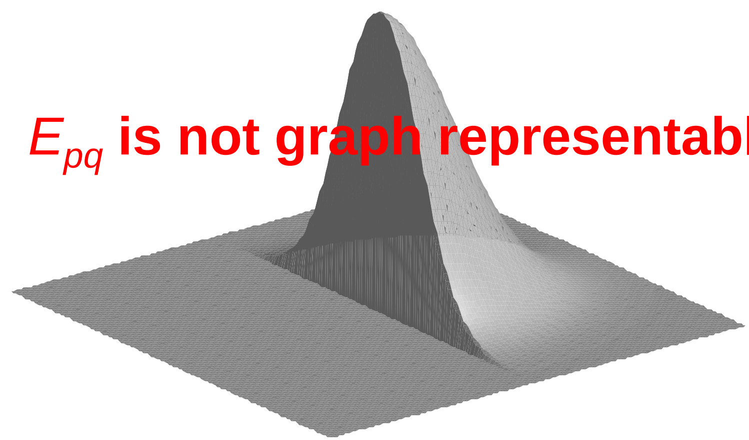
$$U_{pq}(x) = (x)^2$$



$$U_{pq}(x) = |x|$$



$$U_{pq}(x) = \begin{cases} x^2 & |x| \leq \pi \\ \pi^2 |x/\pi|^{0.5} & |x| > \pi \end{cases}$$



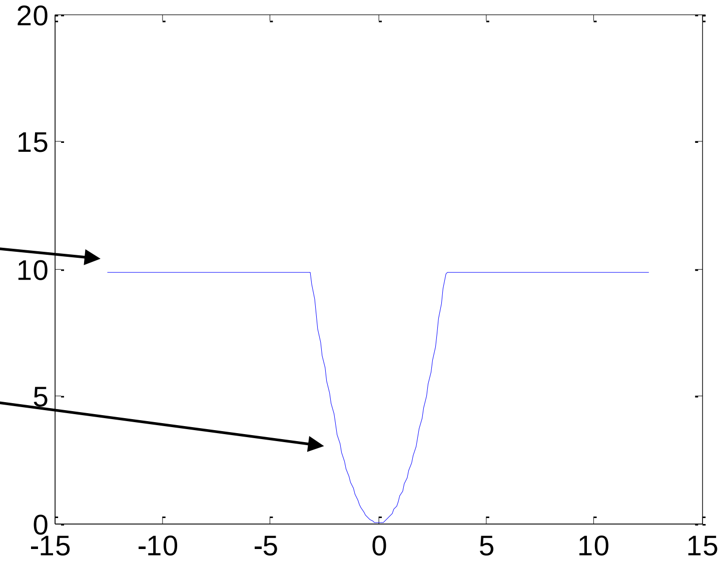
E_{pq} is not graph representable

PUMA: non-convex priors

Ex: $U_{pq}(x) = \min(x^2, \pi^2)$

Models discontinuities

Models Gaussian noise



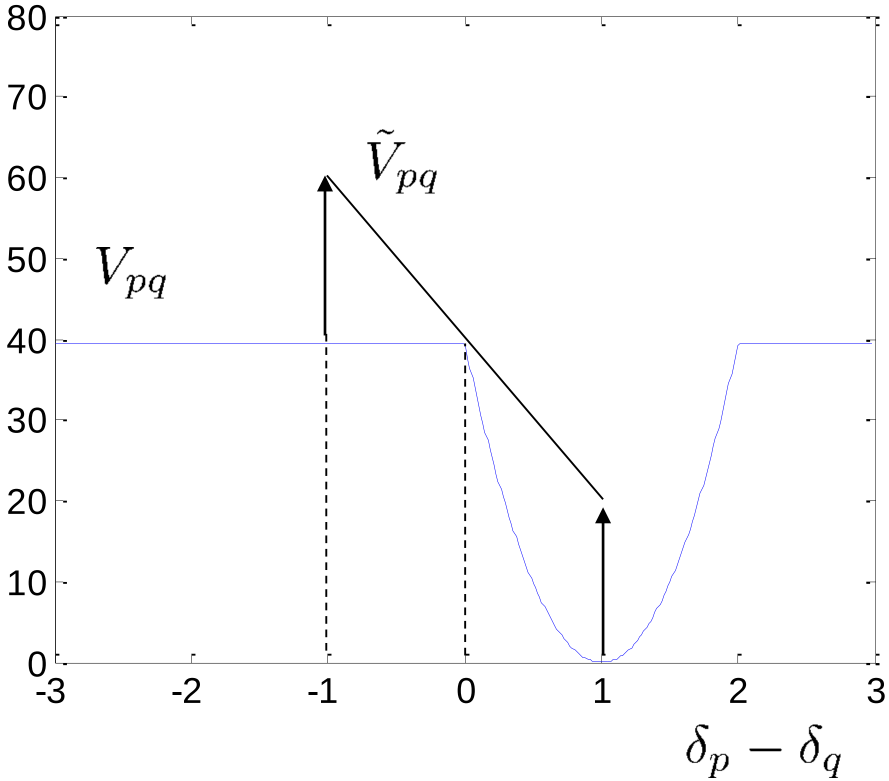
Shortcomings

- ❑ Local minima are no more global minima
- ❑ Energy contains nonsubmodular terms (NP-hard)

Proposed suboptimal solution: majorization minimization applied PUMA binary sub-problems

Majorizing nonsubmodular terms

Majorization Minimization (MM) [Lange & Fessler, 95]



$$\begin{cases} \tilde{U}(\mathbf{k}) = U(\mathbf{k}) \\ \tilde{U}(\mathbf{k}+\boldsymbol{\delta}) \geq U(\mathbf{k}+\boldsymbol{\delta}) \end{cases}$$

$$\boldsymbol{\delta}' = \arg\min_{\boldsymbol{\delta}} \tilde{U}(\mathbf{k}+\boldsymbol{\delta})$$

Non-increasing property

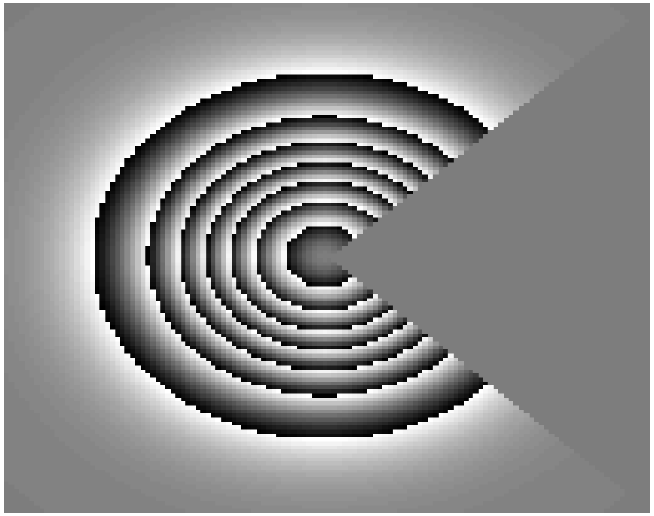
$$U(\mathbf{k}+\boldsymbol{\delta}') \leq U(\mathbf{k})$$

Other suboptimal approaches

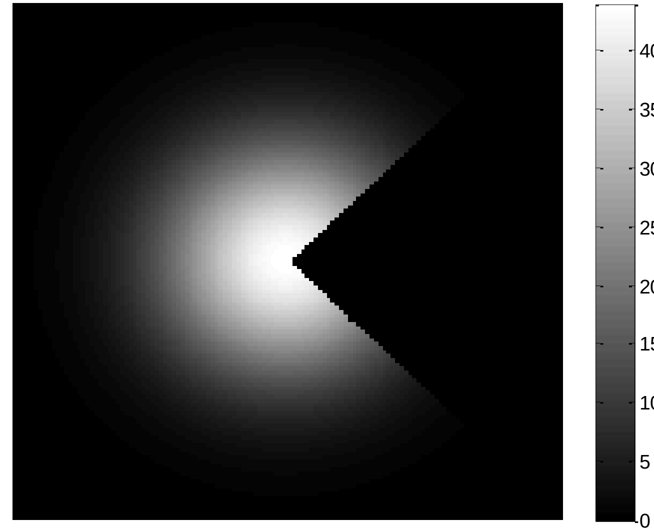
- Quadratic Pseudo Boolean Optimization (Probing [Boros et *al.*, 2006], Improving [Rother et *al.*, 2007])
- Sequencial Tree-Reweighted Message Passing (TRW-S) [Kolmogorov, 2006]
- Dual decomposition (DD) [Komodakis et al., 2011]
- DD + Augmented Lagrangian [Martins et al., 2015]

Results with PUMA (MM) $(n = 128 \times 128, 2^{nd}$ order neighborhood, $p = 0.2, th = 0.1)$

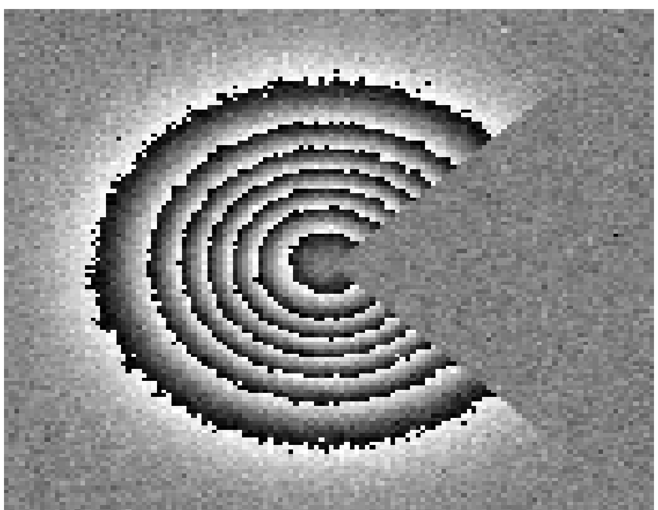
$\boldsymbol{\eta}$



$\widehat{\phi}(\mathrm{MM})$ (8 iter)

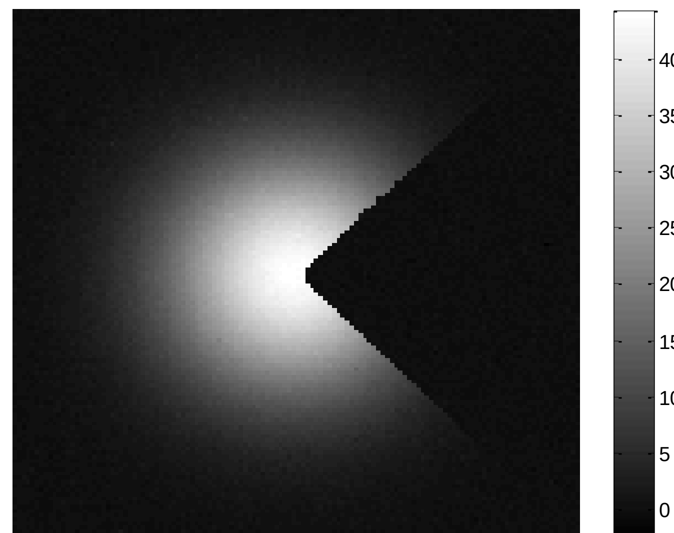


$\boldsymbol{\eta}$



Time = 1s

$\widehat{\phi}(\mathrm{MM})$ (8 iter)



PUMA/IRTV in a HDRP example $\phi \in [0, \rho]$ $n = 256 \times 256$

PUMA: 1^{st} order neighborhood, $p = 0.2$ $th = 0.1$

$\phi \bmod 2\pi$ $(\rho = 6)$

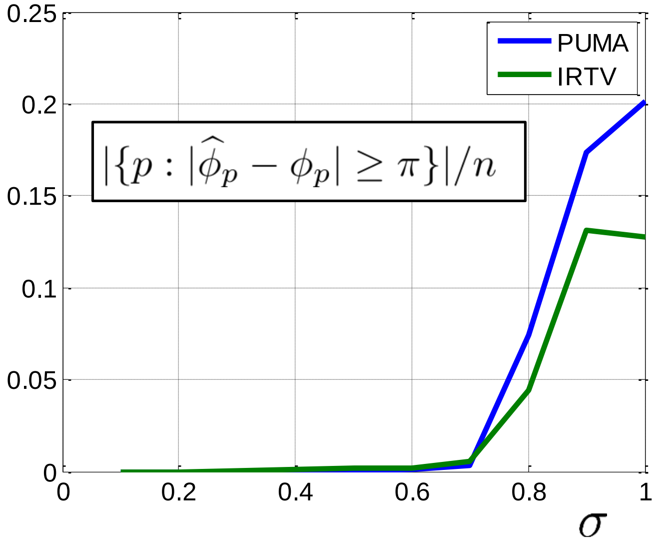
$\phi \bmod 2\pi$ $(\rho = 8)$

ρ	SNR (dB) PUMA	SNR (dB) IRTV
4	∞	∞
5	∞	25.65
6	25.2	19.98
7	17.34	16.09
8	13.68	0.92
9	1.82	2.17
T(sec)	1	350

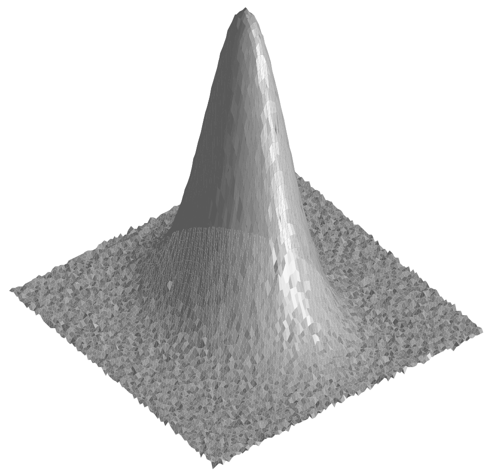
[Kamilov et al.,15]

Degradation mechanisms: noise + "phase discontinuities"

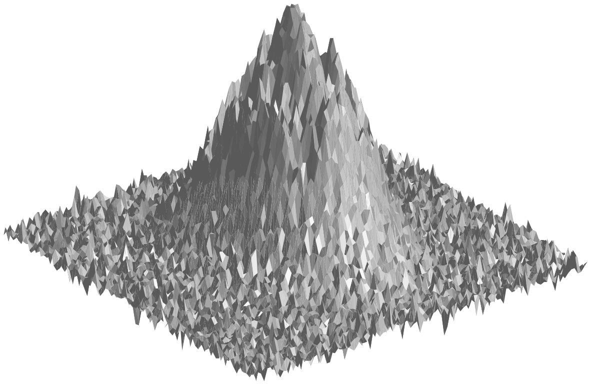
"Phase wraps"



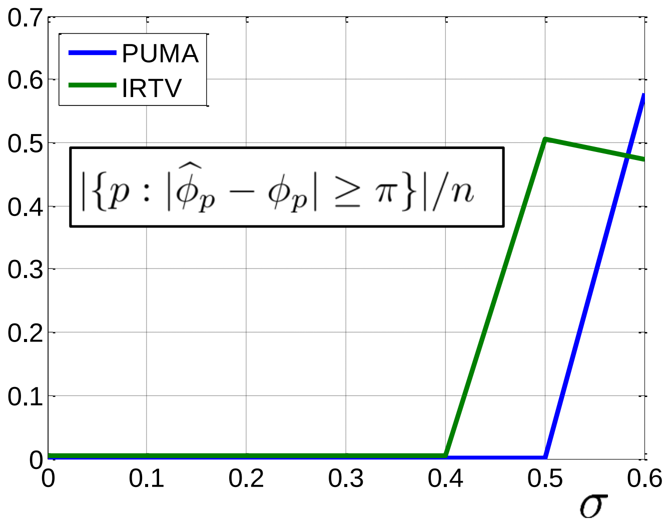
$\sigma = 0.4$



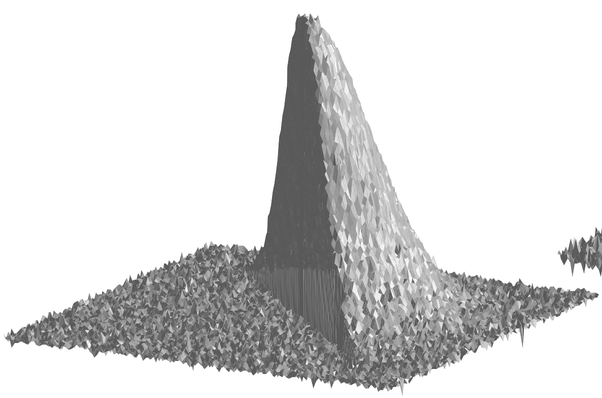
$\sigma = 1.0$



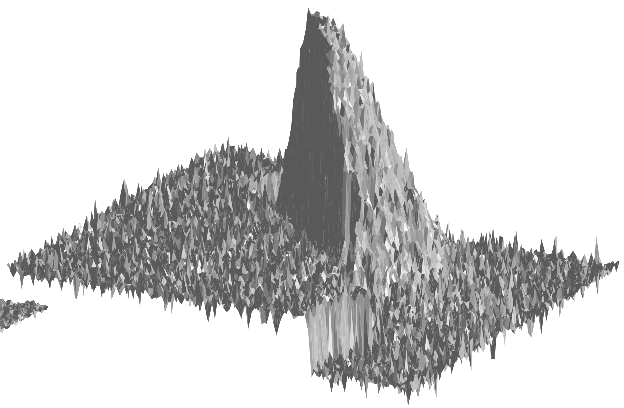
"Phase wraps"



$\sigma = 0.3$



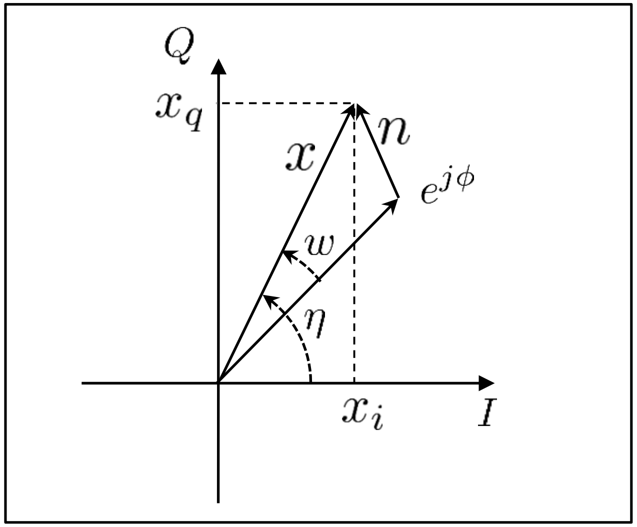
$\sigma = 0.6$



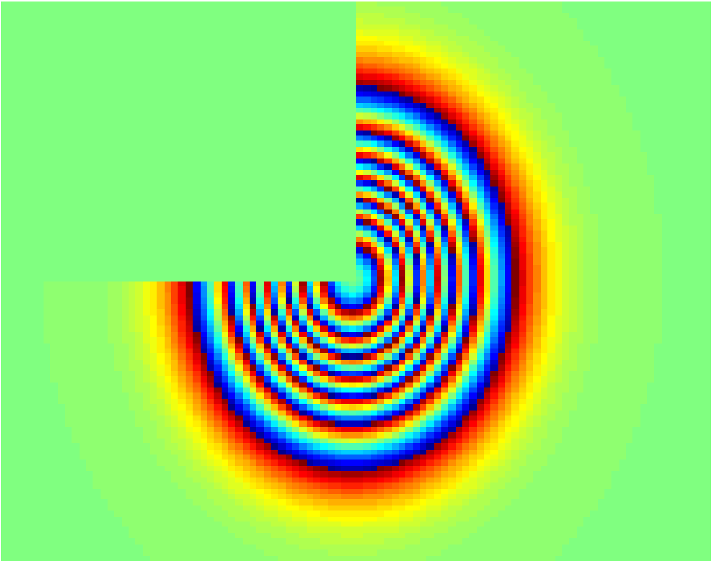
Interferometric phase denoising

objective: estimate $\mathcal{W}[\phi]$ from η

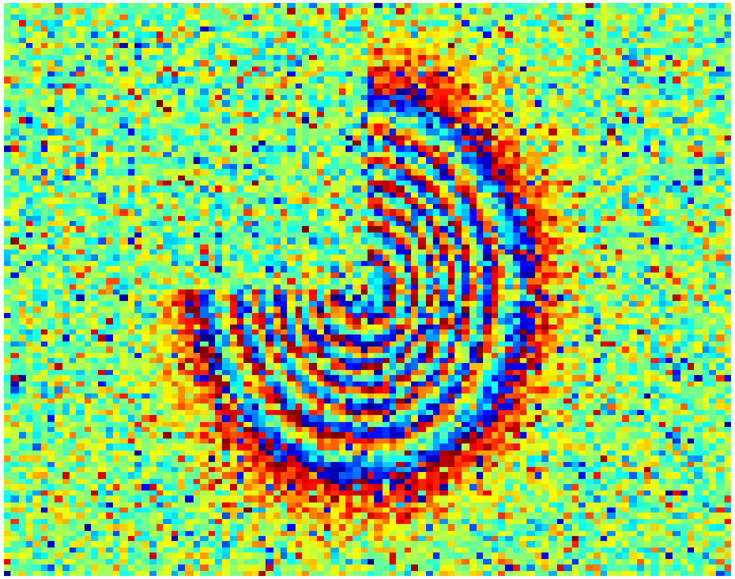
phase modulo 2π



original interf. image $\phi_{2\pi} \equiv \mathcal{W}[\phi]$



observed interf. image η



State-of-the-art in interferometric phase estimation

$$x = ae^{j\phi} + n$$

- Unwrap (first) + denoise

  CAPE [Valadao & B-D, 09]: unwrap with PUMA and then minimize $E(\phi_\pi, \mathbf{k})$ w.r.t. ϕ_π

- parametric model for ϕ

  PEARLS [B-D et al., 2008]: local first order approximation for phase and adaptive window selection (ICI [Katkovnik et al., 06])

- denoise $\mathbf{x}$

  WFT [Kemao, 2007]: windowed Fourier thresholding

- non-local means filtering

  NL-InSAR/NL-SAR [Deledalle, et al., 11, 15]: patch similarity criterion suitable to SAR images and a weighted maximum likelihood estimation interferogram with weights derived in a data-driven way.

Dictionary based interferometric phase estimation

Motivation

- sparse and redundant representations are at the heart of many state-of-the-art applications namely in image restoration
- phase images exhibit a high level of self-similarity. So they admit sparse representations on suitable dictionaries.

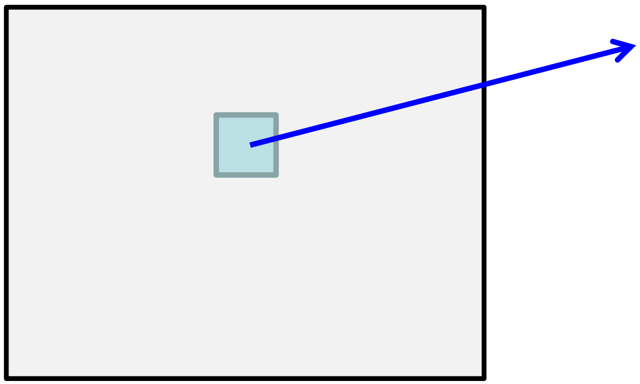
Challenge: the observation mechanism linking the observed phase η with the interferometric phase $\phi_{2\pi}$ is nonlinear.

Observation: the fact that the amplitude and phase images $\mathbf{a}$ and $\boldsymbol{\phi}$ are self-similar, implies that $\mathbf{a}e^{j\boldsymbol{\phi}}$ is self-similar

Our approach: learn sparse representations for $\mathbf{a}e^{j\boldsymbol{\phi}}$ and from them infer $\mathbf{a}$ and $\boldsymbol{\phi}$

Interferometric Phase Estimation via Sparse Regression

Complex valued image



patch of size $\sqrt{m} \times \sqrt{m}$ at pixel *i*

$$\mathbf{x}_i = \mathbf{z}_i + \mathbf{n}_i \in \mathbb{C}^m$$

noise vector

original vector

observed vector

$\mathbf{D} \equiv [\mathbf{d}_1, \ldots, \mathbf{d}_k] \in \mathbb{C}^{m \times k}$ dictionary with respect to which $\mathbf{z}_i$ admits a sparse representation

$$\widehat{\mathbf{z}}_i = \mathbf{D}\widehat{\boldsymbol{\alpha}}_i \qquad \min_{\boldsymbol{\alpha}} \|\boldsymbol{\alpha}\|_0, \quad \text{s.t.:} \quad \|\mathbf{D}\boldsymbol{\alpha} - \mathbf{x}_i\|_2^2 \leq \delta$$

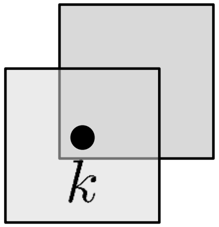
estimation error $\boldsymbol{\varepsilon}_i = \widehat{\mathbf{x}}_i - \mathbf{x}_i$

i.i.d. noise $\Rightarrow$

$$\frac{\|\boldsymbol{\varepsilon}_i\|_2^2}{\|\mathbf{n}_i\|_2^2} \simeq \frac{p}{m}$$

$$p = \|\widehat{\boldsymbol{\alpha}}\|_0$$

Interferometric phase estimation



$\mathcal{P}_k \rightarrow$ the set of patches containing the pixel k

$\widehat{z}_i = z_i + \varepsilon_i, \quad i \in \mathcal{P}_k$ the set of estimates of z_k obtained from patches $i \in \mathcal{P}_k$

Maximum likelihood estimate of $z_i = ae^{j\phi}$

(assume that $\boldsymbol{\varepsilon}_i = [\varepsilon_1, \ldots, \varepsilon_p]$ is $\mathcal{N}(\mathbf{0}, \mathbf{C})$)

$$\widehat{\phi}_{2\pi} = \arg\left(\sum_{j=1}^{q} \widehat{z}_j \gamma_j\right) \qquad \widehat{a} = \frac{\left|\sum_{j=1}^{q} \widehat{z}_j \gamma_j\right|}{\sum_{j=1}^{q} \gamma_j}$$

where $\gamma_j := \sum_{k=1}^{q} [\mathbf{C}^{-1}]_{jk}$.

In practice γ_j is very hard to compute and we take $\gamma_j = c^{te}$

Dictionary learning

Find a dictionary representing accurately the image patches with the smallest possible number of atoms.

formalization under the regularization framework

$$\min_{\mathbf{D}\in\mathcal{C},\mathbf{A}} L(\mathbf{D},\mathbf{A}) \qquad L(\mathbf{D},\mathbf{A}) = (1/2)\big\|\mathbf{X}-\mathbf{D}\mathbf{A}\big\|_F^2 + \lambda\|\mathbf{A}\|_1,$$

where $\mathcal{C} := \left\{\mathbf{D}\in\mathbb{C}^{m\times k} \ : \ \left|\mathbf{d}_j^H\mathbf{d}_j\right| \le 1,\ j=1,\dots,k\right\}$

and $\mathbf{X} = [\mathbf{x}_1,\dots,\mathbf{x}_{N_p}]$ and $\mathbf{A} = [\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_{N_p}]$

DL Algorithm: alternating proximal minimization (APM)

$$\begin{cases} \mathbf{D}^{k+1} \in \arg\min\limits_{\mathbf{D}\in\mathcal{C}} L(\mathbf{D},\mathbf{A}^k) + \lambda\|\mathbf{D}-\mathbf{D}^k\|_F^2 \\ \mathbf{A}^{k+1} \in \arg\min\limits_{\mathbf{A}} L(\mathbf{D}^{k+1},\mathbf{A}) + \lambda\|\mathbf{A}-\mathbf{A}^k\|_F^2 \end{cases}$$

Convergence (based on the Kurdyka- Lojasiewicz inequality)
[Attouch et al. 10], [Xu, Yin, 2012]

Dictionary learning

drawback: alternating proximal minimization takes too long (order of 10^4 sec) in a typical image scenario (N_p = 100000, m = 100, and k = 200)

Online Dictionary Learning (ODL): [Mairal et al. 2010]

Select randomly $\mathbf{x}^t \equiv [\mathbf{x}_i^t \; i = 1, \dots \eta]$ from $\mathbf{z}$

(*Sparse coding:* BPDN)

$$\boldsymbol{\alpha}^t := \arg \min_{\boldsymbol{\alpha} \in \mathbb{C}^{k \times \eta}} (1/2) \|\mathbf{x}^t - \mathbf{D}\boldsymbol{\alpha}\|_F^2 + \lambda \|\boldsymbol{\alpha}\|_1$$

$$\min_{\mathbf{D} \in \mathcal{C}} \frac{1}{S_t} \sum_{i=1}^{t} w_i \left\{ (1/2) \left\|\mathbf{x}^i - \mathbf{D}\boldsymbol{\alpha}^i\right\|_F^2 + \lambda \|\boldsymbol{\alpha}^i\|_1 \right\}$$

$\mathbf{D}^t$ converges to the stationary points of

$$(1/2)\left\|\mathbf{X} - \mathbf{DA}\right\|_F^2 + \lambda \|\mathbf{A}\|_1,$$

$$\mathbf{D} \in \mathcal{C}$$

Computational complexity: $O(km^2 + \eta k m)$

The proposed denoising algorithm

SpInPHASE [Hongxing, B-D, Katkovnik, 14]

Input: $\mathbf{x} \in \mathbb{C}^{N_1 \times N_2}$ (complex valued image)

Ouput: $\widehat{\phi} \in \mathbb{R}^{N_1 \times N_2}$ (absolute phase estimate)

Begin

$\mathbf{x}_i \leftarrow \mathbf{M}_i\mathbf{x}, \;\; i = \;\ldots, N_p$ (extract patches)

$\mathbf{D} \leftarrow \mathrm{DL}(\mathbf{x}_i, i = 1, \ldots, N_p)$ (learn the dictionary)

$\boldsymbol{\alpha}_i \leftarrow \mathrm{OMP}(\mathbf{D}, \mathbf{x}_i, i = 1, \ldots, N_p)$ (sparse coding)

$\widehat{\mathbf{z}}_i \leftarrow \mathbf{D}\boldsymbol{\alpha}_i, i = 1, \ldots, N_p)$ (patch estimate)

$\widehat{\mathbf{x}} \leftarrow \mathrm{compose}(\widehat{\mathbf{z}}_i, i = 1, \ldots, N_p)$ (patch compose)

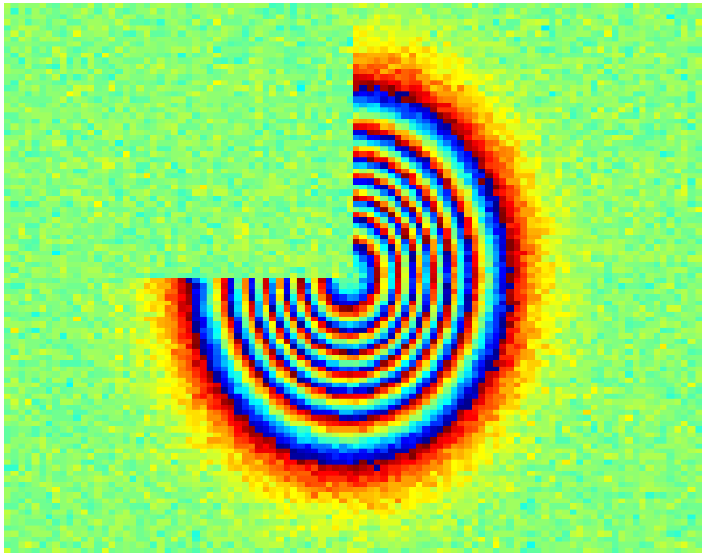
$\widehat{\phi}_{2\pi} \leftarrow \arg(\widehat{\mathbf{x}})$ (interferometric phase estimate)

$\widehat{\phi} \leftarrow \mathrm{PUMA}(\widehat{\phi}_{2\pi})$ (phase unwrapping)

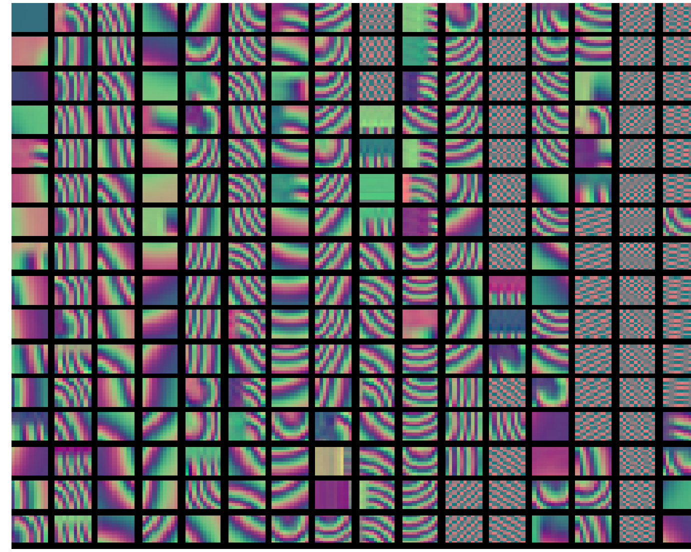
End

DL: Example

(truncated Gaussian - $\sigma = 0.3$) $\sqrt{m} = 12,\ k = 256$



learned dictionary

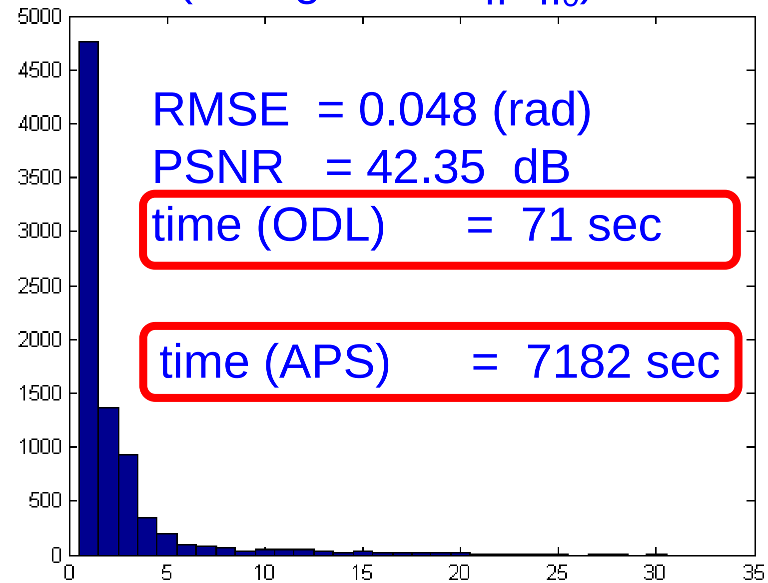


$$\text{RMSE} := \frac{\|\mathcal{W}(\widehat{\phi}_{2\pi} - \phi_{2\pi})\|_F}{\sqrt{N}}$$

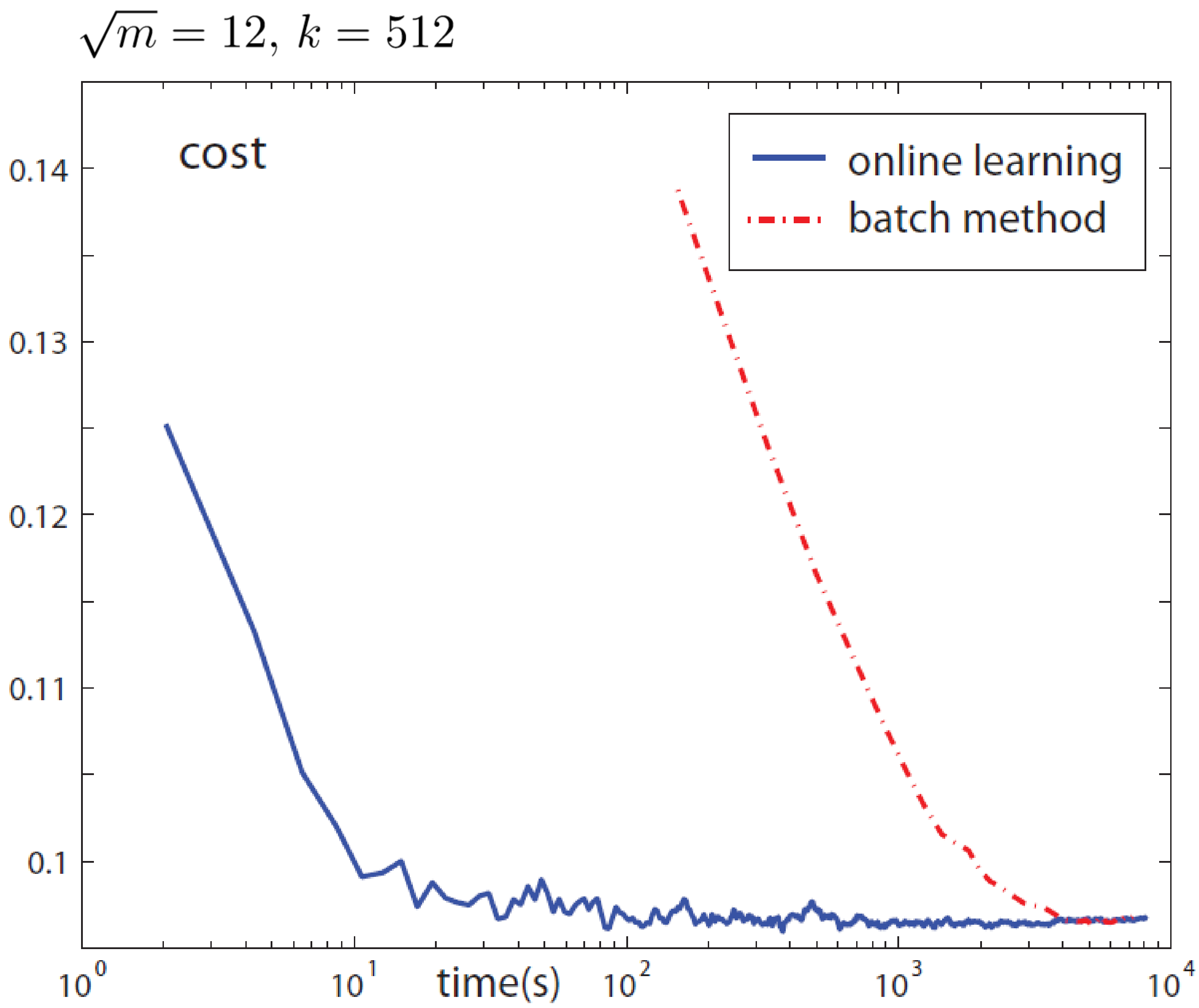
$$\text{PSNR} := \frac{4N\pi^2}{\|\mathcal{W}(\widehat{\phi}_{2\pi} - \phi_{2\pi})\|_F^2}$$

$$\frac{\|\mathcal{W}(\boldsymbol{\eta} - \phi_{2\pi})\|_F^2}{\|\mathcal{W}(\widehat{\phi}_{2\pi} - \phi_{2\pi})\|_F^2} = 20 \simeq \frac{1}{2}\frac{m}{p}$$

(histogram of $\|\alpha\|_0$)

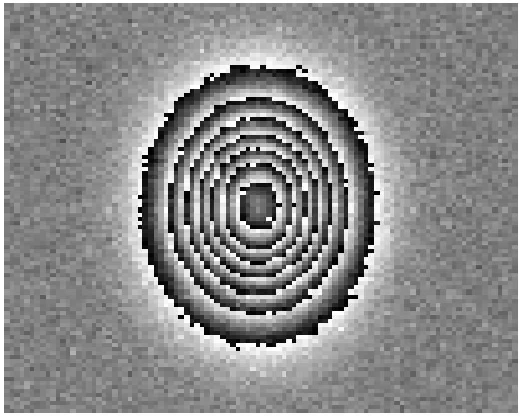


DL: Online (ODL) Versus Batch (APM)

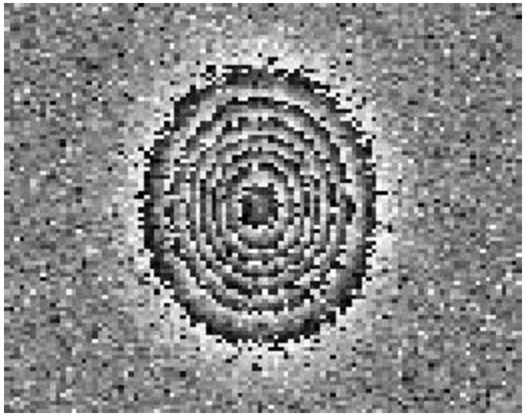


Restored Images

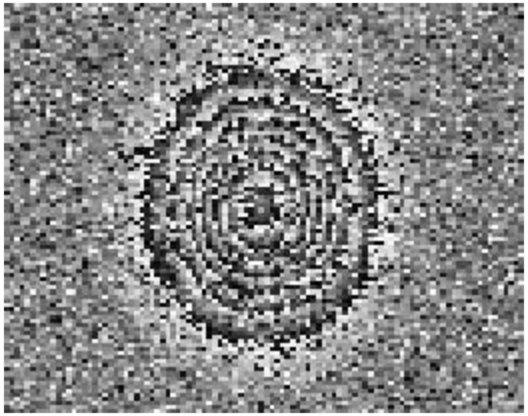
$\sigma = 0.5$

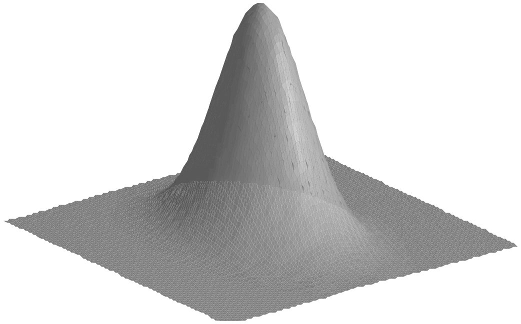


$\sigma = 1.0$

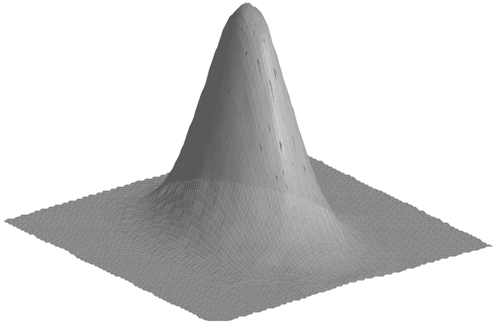


$\sigma = 1.5$

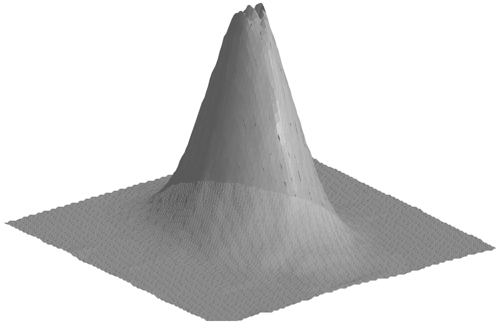




RMSE = 0.052

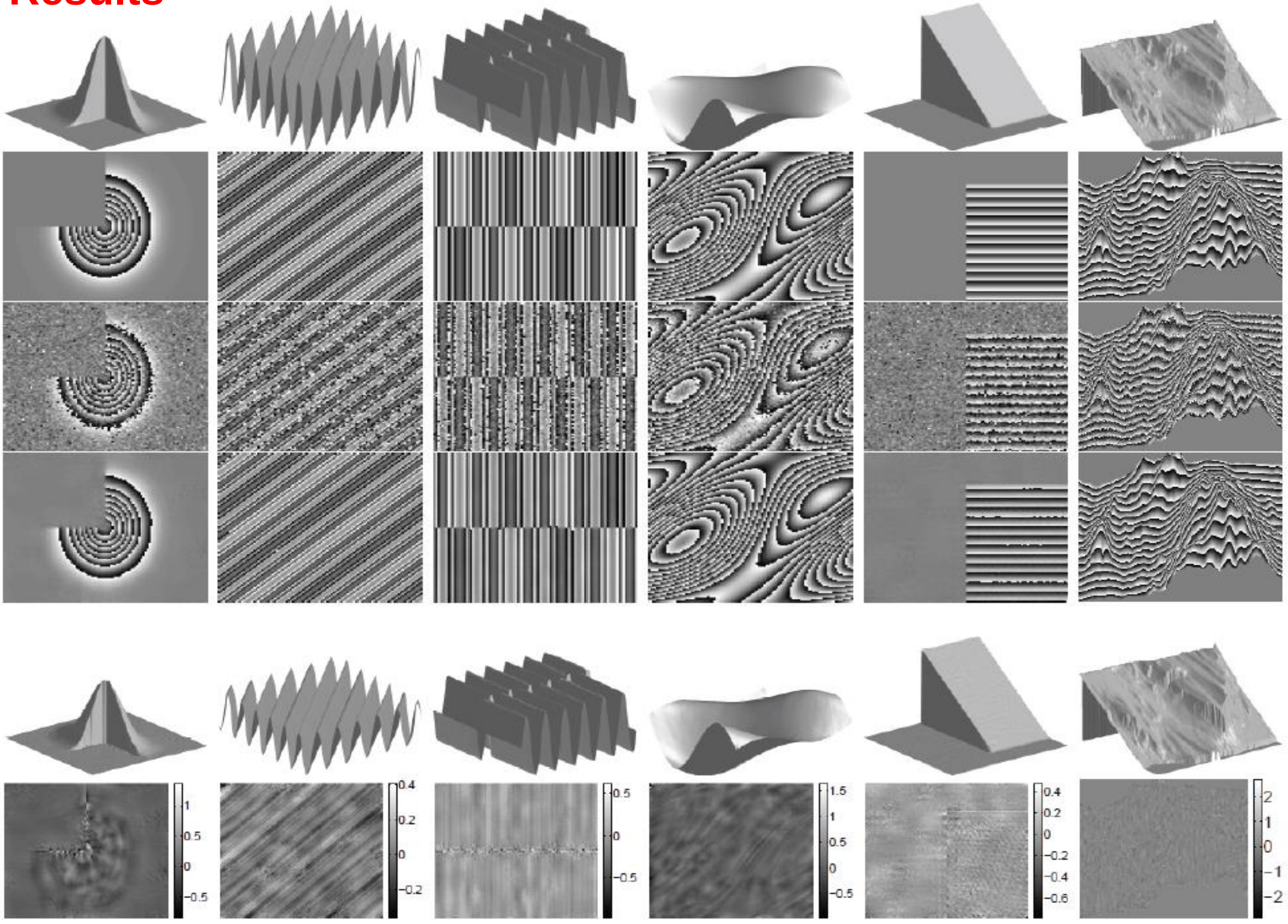


RMSE = 0.108



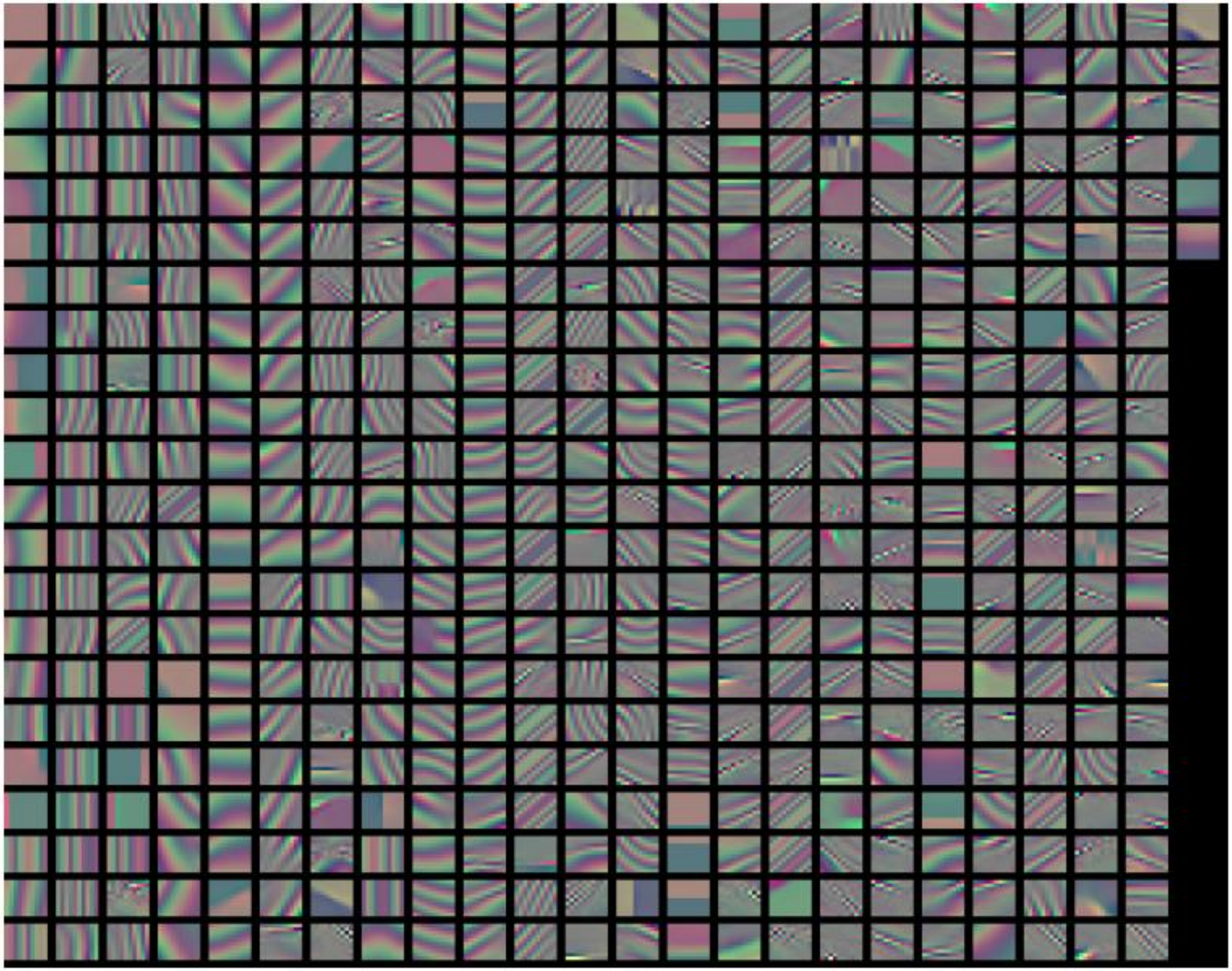
RMSE = 0.174

Results



Dictionary learned from 6 images (shown before)

$\sqrt{m} = 12,\ k = 512$



Comparisons with competitors

Surf.	σ	PSNR (dB) Sp(ld)	PSNR (dB) Sp(pd)	PSNR (dB) W	PSNR$_a$ (dB) Sp(ld)	PSNR$_a$ (dB) Sp(pd)	PSNR$_a$ (dB) W	NELP Sp(ld)	NELP Sp(pd)	NELP W	TIME (s) Sp(ld)	TIME (s) Sp(pd)	TIME (s) W
Trunc. Gauss.	0.3	42.51	**42.88**	40.29	42.51	**42.88**	40.29	**0**	**0**	**0**	69	**6**	10
	0.5	39.63	**39.95**	36.71	39.63	**39.95**	36.71	**0**	**0**	**0**	74	**4**	10
	0.7	35.69	**36.96**	34.26	35.85	**36.98**	34.37	8	**3**	10	72	**3**	10
	0.9	33.52	**36.04**	32.79	33.52	**36.23**	32.79	**0**	7	**0**	72	**3**	10
Sinu.	0.3	**48.94**	47.77	35.76	**48.94**	47.77	35.76	**0**	**0**	0	61	**2**	10
	0.5	41.91	**43.50**	31.48	41.91	**43.50**	31.48	**0**	**0**	**0**	65	**2**	10
	0.7	38.44	**41.20**	28.90	38.44	**41.20**	28.90	**0**	**0**	**0**	65	**2**	10
	0.9	36.42	**39.30**	26.36	36.42	**39.30**	26.36	**0**	**0**	**0**	63	**2**	10
Sinu. discon.	0.3	**44.45**	42.29	35.91	**44.45**	42.29	35.91	**0**	**0**	**0**	63	**6**	10
	0.5	**39.41**	38.61	31.86	**39.41**	38.61	31.86	**0**	**0**	**0**	72	**3**	10
	0.7	**37.09**	35.95	29.86	**37.09**	35.95	29.95	**0**	**0**	1	71	**2**	10
	0.9	**34.17**	34.00	27.64	**34.17**	34.00	27.71	**0**	**0**	6	66	**2**	10
Mount.	0.3	**40.66**	38.90	40.00	**40.66**	38.90	40.00	**0**	**0**	**0**	57	**10**	**10**
	0.5	**37.20**	35.66	36.55	**37.20**	35.66	36.55	**0**	**0**	**0**	60	**6**	10
	0.7	**34.35**	33.29	34.17	**34.35**	33.29	34.17	**0**	**0**	**0**	62	**5**	10
	0.9	**32.55**	31.66	32.31	**32.70**	31.79	32.31	1	1	**0**	60	**4**	10
Shear plane	0.3	**49.36**	47.01	40.67	**49.36**	47.01	40.67	0	**0**	**0**	57	23	**10**
	0.5	**42.95**	44.05	37.07	**42.95**	44.05	37.07	**0**	**0**	**0**	63	**2**	10
	0.7	38.39	**39.58**	34.13	38.39	**39.58**	34.13	**0**	**0**	**0**	68	**2**	10
	0.9	33.53	**38.72**	33.24	33.53	**38.72**	33.24	**0**	**0**	**0**	72	**2**	10
Long's Peak	0.3	35.49	**35.68**	35.40	35.51	**35.69**	35.41	**28**	**28**	**28**	515	179	**31**
	0.5	33.05	**33.19**	32.89	33.08	**33.24**	32.93	32	33	**31**	357	77	**30**
	0.7	31.32	**31.46**	31.19	31.46	**31.53**	31.28	**26**	48	32	326	42	**30**
	0.9	29.97	**30.17**	29.90	30.09	**30.26**	29.99	34	**32**	35	308	**27**	30

Concluding remarks

- Overview absolute phase estimation, from interferometric measurements, based a linear observation formulation and on phase unwarpping

- The need for interferometric phase estimation

- SpInPhase: Interferometric phase denoising via sparse coding in the complex domain

  - Exploits the self-similarity of the complex valued images
  - State-of-the-art results, namely regarding the preservation of discontinuities coded in the interferometric phase $e^{j\phi}$

- Current research directions

  - Multisource phase estimation
  - Denoising via sparse coding in the complex domain via high-order SVD and nonlocal block matching techniques
  - Phase retrieval with patch-oriented dictionaries

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Multi-source absolute phase estimation

Ex: different frequencies $p_1(z_i|\phi) \propto c_i\, e^{\lambda_i \cos(f_i\phi - \eta_i)}$

Two sources. Ex: $f_1 = 1,\ f_2 = \frac{u}{v} \qquad u, v \in \mathbb{N} \qquad \text{primes}$

$$d(\phi) = -\lambda_1 \cos(\phi - \eta_1) - \lambda_2 \cos(f_2\phi - \eta_2)$$

$$d(\phi + 2\pi v) = d(\phi) \ \Rightarrow 2v\pi\text{-periodic}$$

LOM formulation $\mathbf{y} = \mathbf{D}\boldsymbol{\phi} + \mathbf{w}_\eta + \mathbf{w}_\pi$

$$\eta \in \arg \min_{[-v\pi, v\pi[^n} \sum_{p\in\mathcal{V}} d_p(\phi)$$

$$\mathbf{y} = \mathcal{W}_v(\mathbf{D}\boldsymbol{\eta})$$

Integer formulation: unwrap phase images with range larger than $2v\pi$

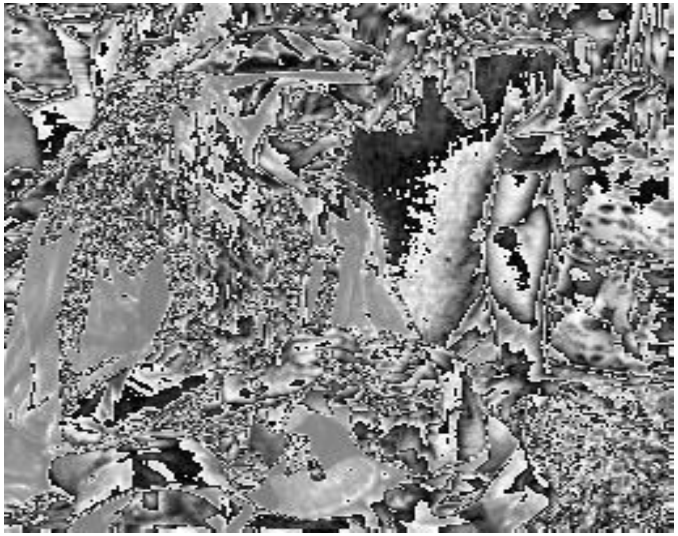
$$\min_{\mathbf{k}\in\mathbb{Z}^n} \sum_{\{p,q\}\in\mathcal{E}} V_{pq}(k_p - k_q)$$

$$V_{pq}(k_p - k_q) = U_{pq}(\eta_p - \eta_q + 2\pi v(k_p - k_q))$$

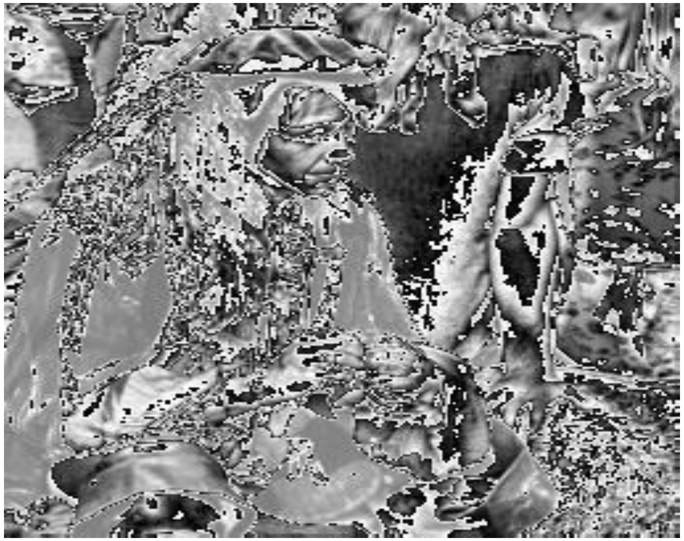
Noise is an issue

Example: two sources, image man $\rho = 10\pi \qquad f_1 = 1, f_2 = \frac{3}{4} \;\Rightarrow v = 4$

$\mathcal{W}_\pi(\mathbf{x}_1)$



$\mathcal{W}_\pi(\mathbf{x}_2)$

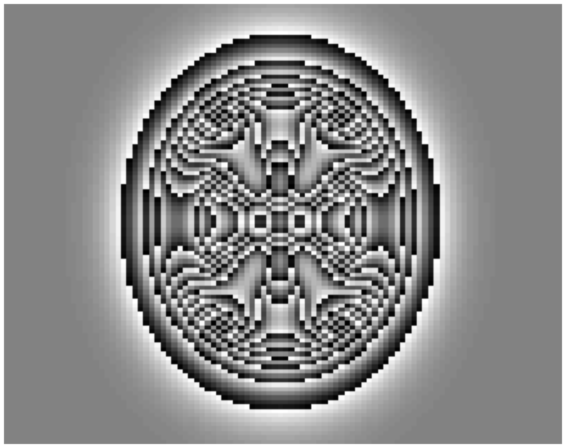


$\mathrm{SNR} = 58\,\mathrm{dB}$

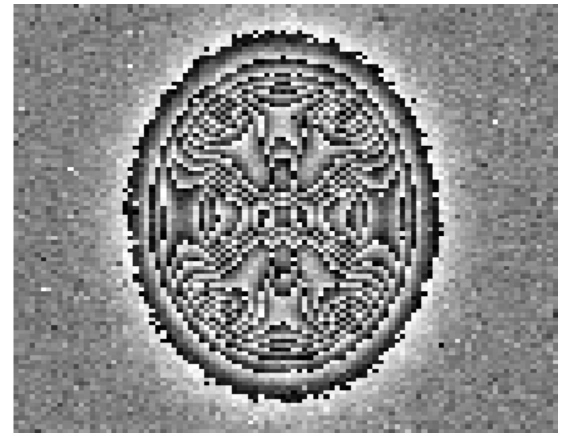
$$\boldsymbol{\eta} = \arg\min_{\phi} -\lambda_1 \cos(\phi - \eta_1) - \lambda_2 \cos(f_2 \phi - \eta_2) \qquad f_2 = \frac{2}{3}$$

---

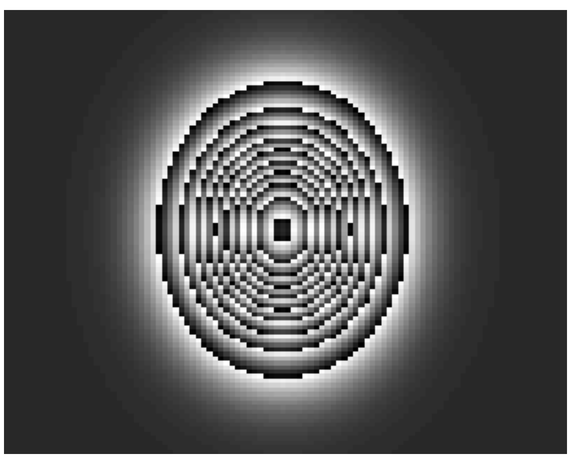
phase range $= 60\pi$



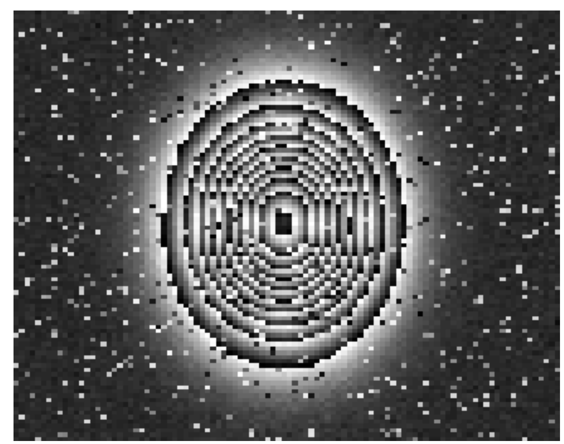
SNR = 5 dB

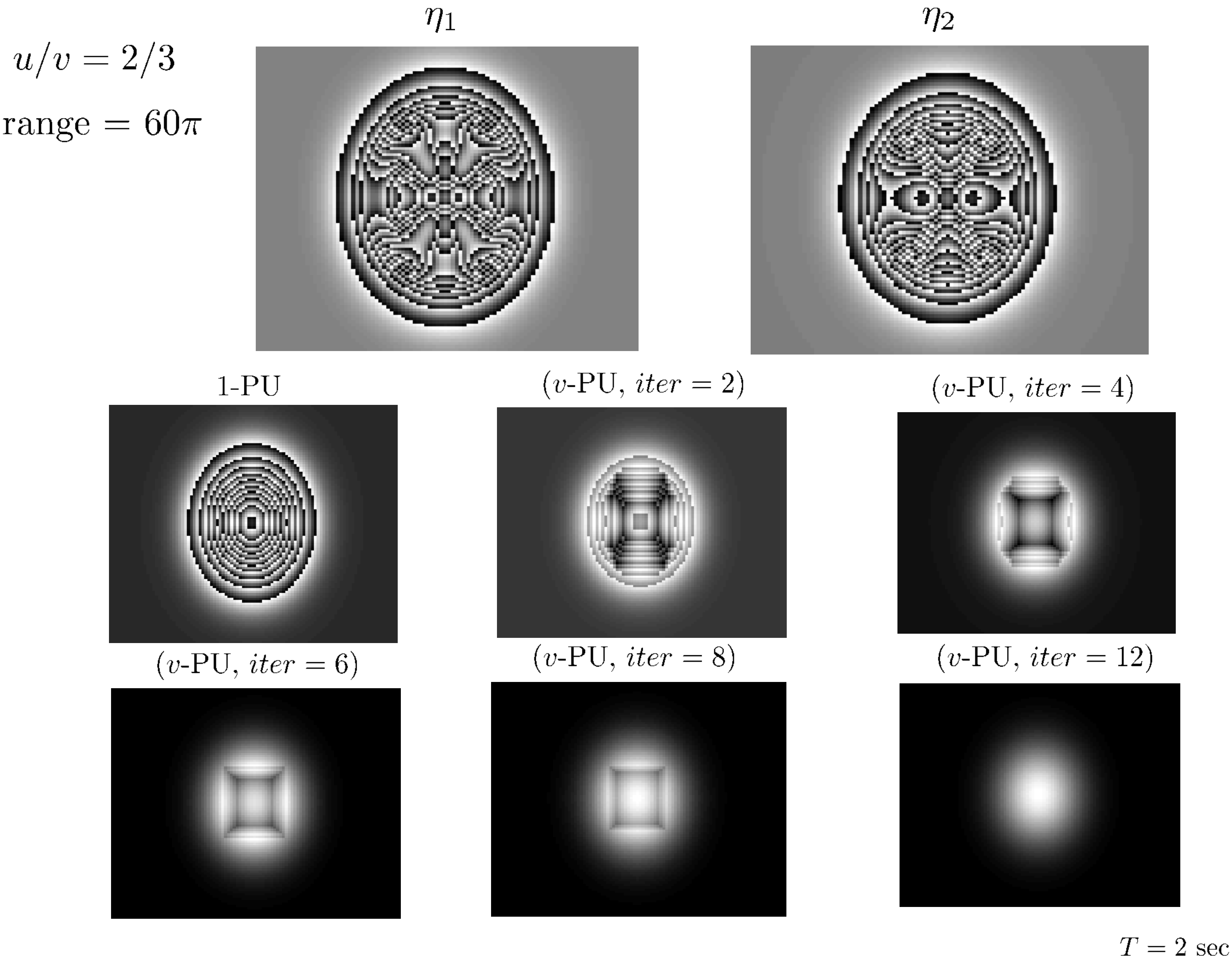


$\boldsymbol{\eta}$



$\boldsymbol{\eta}$




u/v = 2/3
range = 60π
η1
η2
1-PU
(v-PU, iter = 2)
(v-PU, iter = 4)
(v-PU, iter = 6)
(v-PU, iter = 8)
(v-PU, iter = 12)
T = 2 sec

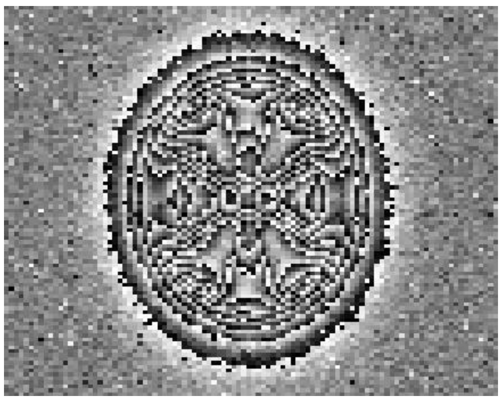
v-Interferometric Phase Estimation via DL

Approach

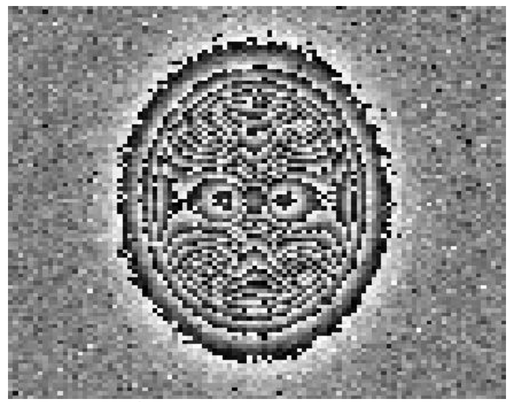
$$\min_{\phi \in [-\pi v, \pi v[^{|\mathcal{V}|}} -\log p(\mathbf{z}|\phi) + \tau_1 \left\| e^{j\frac{\phi}{v}} - L(\mathbf{DA}) \right\|_F^2 + \tau_2 \|\mathbf{A}\|_{1,1}$$

Example $u/v = 2/3$ $\quad$ $\text{range} = 60\pi$

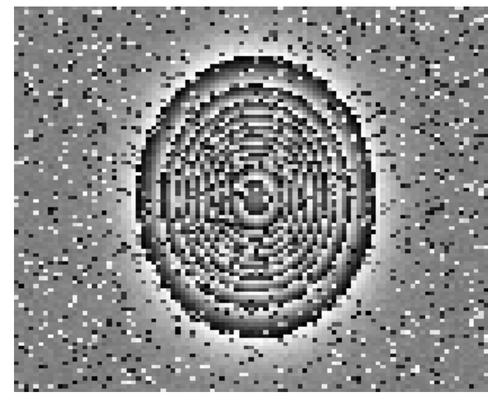
η_1



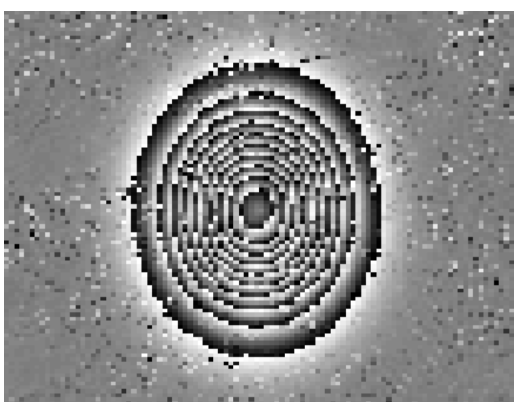
η_2



ψ

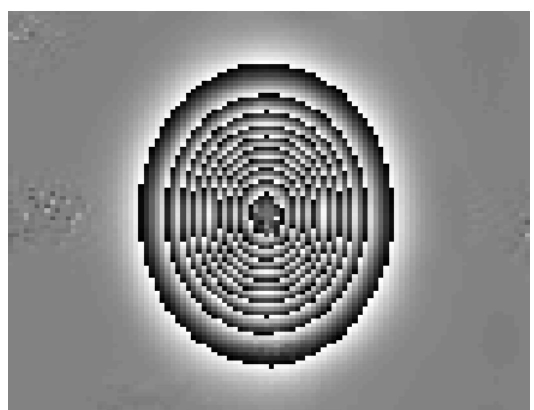


$\widehat{\phi}_{2\pi v}$ (iter -1)



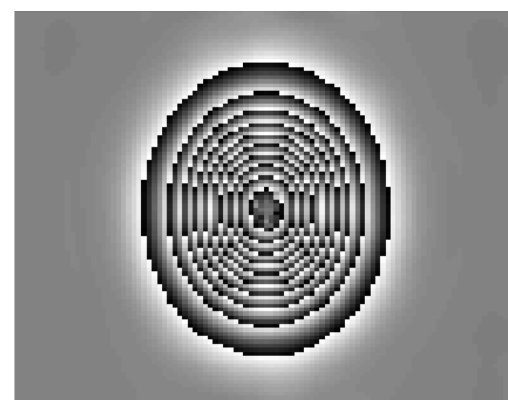
RMSE ($\phi_{2\pi v} = 0.86$)

$\widehat{\phi}_{2\pi v}$ (iter -2)

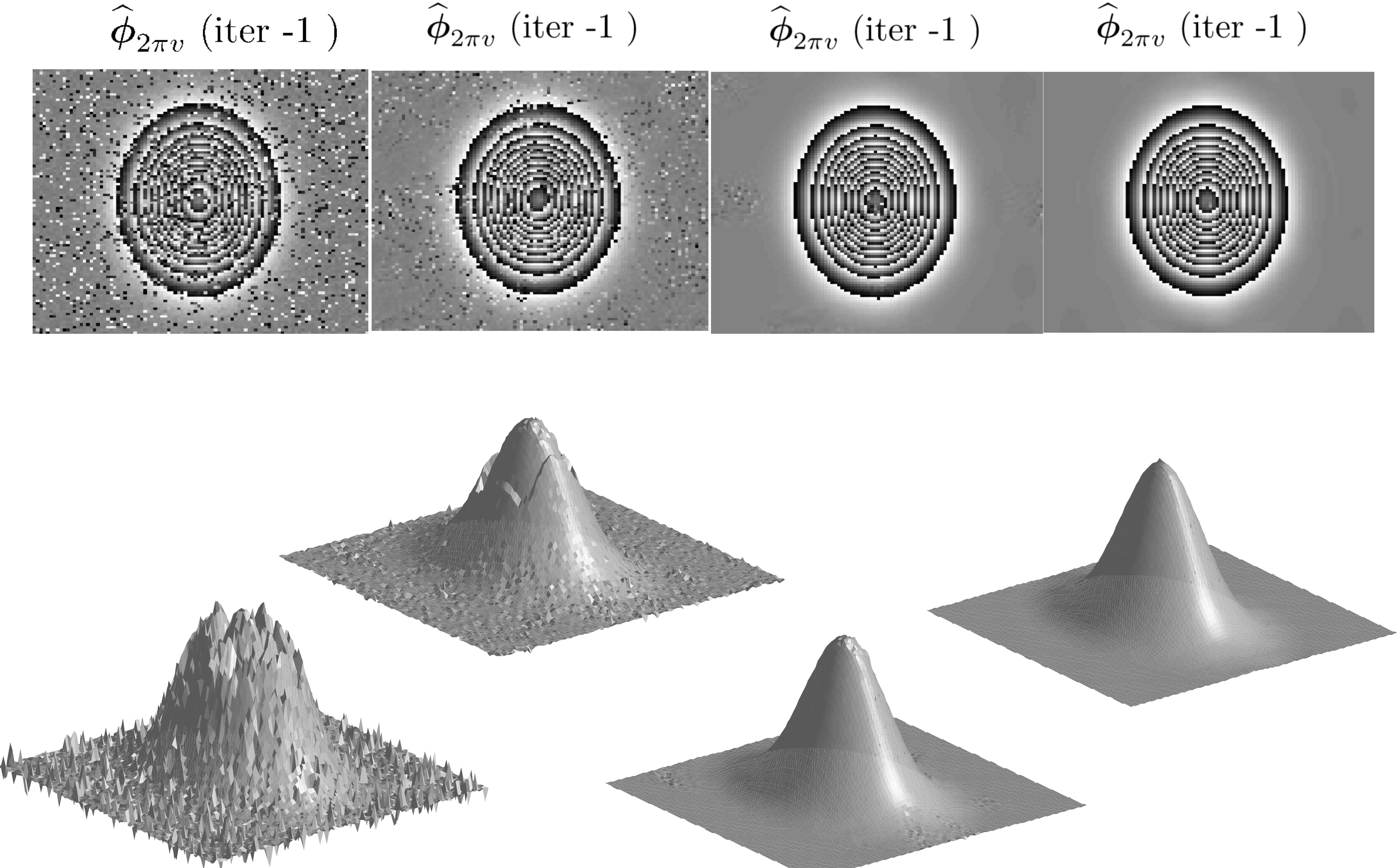


RMSE ($\phi_{2\pi v} = 0.27$)

$\widehat{\phi}_{2\pi v}$ (iter -3)



RMSE ($\phi_{2\pi v} = 0.19$)


$\widehat{\phi}_{2\pi v}$ (iter -1)
$\widehat{\phi}_{2\pi v}$ (iter -1)
$\widehat{\phi}_{2\pi v}$ (iter -1)
$\widehat{\phi}_{2\pi v}$ (iter -1)