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Koszul/Souriau Fisher Metric Spaces & Optimization by Maximum Entropy: Hessian Information Geometry, Lie Group Thermodynamics & Poincaré-Marle-Souriau Equation

Frédéric BARBARESCO, THALES AIR SYSTEMS
Senior Scientist & Advanced Studies Manager
Advanced Radar Concepts Dept.
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Problems to be solved by THALES for Radar Applications

- **Problem 1: How to define density of probability for covariance matrices of stationary time series (THPD matrix: Toeplitz Hermitian Positive Definite matrix)**

$$p(\xi / \bar{\xi}) = ??$$

$$\xi^{+} = \xi \quad \text{Hermitian}$$

$$\det(\xi) > 0, \quad \xi \ \text{Positive Definite}$$

$$\xi \quad \text{Toeplitz}$$

- **Problem 2: How to define « Ordered Statistics » for covariance matrices, knowing that there is no « total orders » for these matrices**

$$\text{Local Order}: \xi_1 < \xi_2 \Rightarrow \xi_2 - \xi_1 > 0 \quad \text{Positive Definite}$$

$$\text{But no Global Order}: \xi_1 \leq \xi_2 \leq .... \leq \xi_n$$

- **Solution to Problem 1: Koszul/Souriau Solution of Maximum Entropy**
  - Entropy as Legendre Transform of Generalized Characteristic Function (Laplace Transform on Convex Cone with Inner Product given by Cartan-Killing form)
  - Density of Probability as Souriau Covariant Solution of Maximum Entropy

$$p_{\bar{\xi}}(\xi) = \frac{e^{-\left\langle \xi, \Theta^{-1}\left(\bar{\xi}\right)\right\rangle}}{\int_{\Omega^*} e^{-\left\langle \xi, \Theta^{-1}\left(\bar{\xi}\right)\right\rangle} d\xi}$$

$$\Phi(x) = -\log \int_{\Omega^*} e^{-\langle x, \xi\rangle} d\xi$$

$$\langle x, y\rangle = Tr\left(ad_x ad_{\theta(y)}\right)$$

$$\bar{\xi} = \int_{\Omega^*} \xi . p_{\bar{\xi}}(\xi) d\xi$$

$$\bar{\xi} = \Theta(x) = \frac{d\Phi(x)}{dx}$$

$$x = \Theta^{-1}(\bar{\xi})$$

- **Solution to Problem 2: Frechet Median Barycenter**
  - Metric given by Koszul Hessian Geometry (hessian of Entropy) & Souriau Lie Group Thermodynamics (metric defined by symplectic cocycle and Geometric temperature)

$$I(x) = -E_{\xi}\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right] = \frac{\partial^2 \log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi}{\partial x^2}$$

  - Geometric Median by Fréchet barycenter in Metric space, solved by Karcher Flow

$$\bar{\xi}_{median} = \underset{\bar{\xi}}{Arg\,Min} \sum_{i=1}^{n} d(\xi_i, \bar{\xi})$$

Koszul-Vinberg Characteristic Function

- **François Massieu in 1869 demonstrated that some thermal properties of physical systems could be derived from "characteristic functions".**
- **This idea was developed by Gibbs and Duhem with the notion of potentials in thermodynamics, and introduced by Poincaré in probability.**
- **We will study generalization of this concept by**
  - Jean-Louis Koszul in Mathematics
  - Jean-Marie Souriau in Statistical Physics.
- **The Koszul-Vinberg Characteristic Function (KVCF) on convex cones will be presented as cornerstone of "Information Geometry" theory:**
  - defining Koszul Entropy as Legendre transform of minus the logarithm of KVCF (their gradients defining mutually inverse diffeomorphisms)
  - Fisher Information Metrics as hessian of these dual functions.
- **Koszul proved that these metrics are invariant by all automorphisms of the convex cones.**

Koszul Characteristic: Massieu/Poincaré/Levy/Balian

$$S = \phi - \frac{1}{T} \cdot \frac{\partial \phi}{\partial (1/T)}$$

1869 **MÉMOIRES PRÉSENTÉS.**

THERMODYNAMIQUE. — *Sur les fonctions caractéristiques des divers fluides.* Mémoire de M. F. Massieu. (Extrait par l'Auteur.)

(Commissaires : MM. Combes, Regnault, Bertrand.)

François MASSIEU

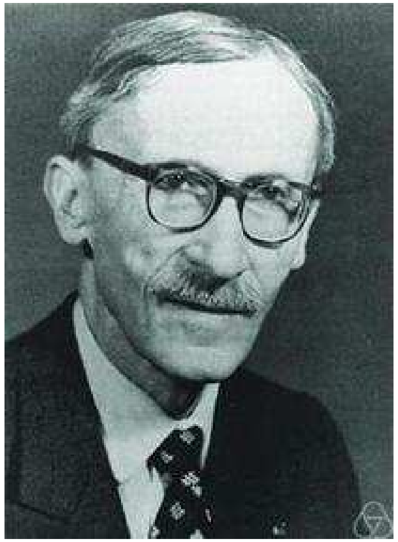
(introduction of characteristic function in Thermodynamic: Gibbs-Duhem Potentials)

« je montre, dans ce mémoire, que toutes les propriétés d'un corps peuvent se déduire d'une fonction unique, que j'appelle la fonction caractéristique de ce corps»

Henri POINCARE

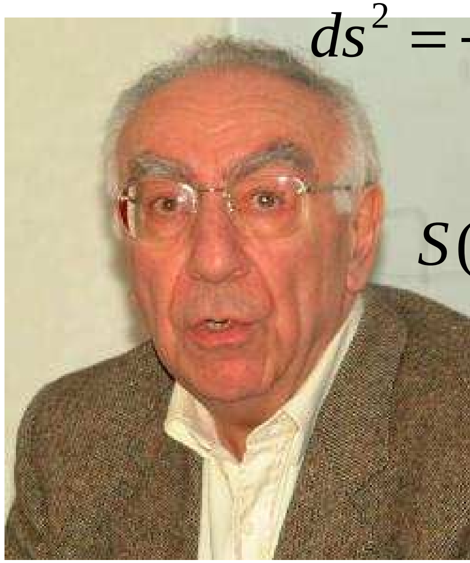
(Introduction of characteristic function Ψ in Probability)

$$\psi = e^{\phi} \quad \text{or} \quad \phi = \log \psi$$



Paul LEVY

(general use of characteristic function in Probability)



$$ds^2 = -d^2 S = Tr\left[d\hat{D} . d \ln \hat{D} \right]$$

$$F(\hat{X}) = \ln Tr \exp \hat{X}$$

$$S(\hat{D}) = F(\hat{X}) - \left\langle \hat{D}, \hat{X} \right\rangle$$

Roger BALIAN

(metric for quantum states by hessian metric from Von-Neumann Entropy)

Roger Balian, 1986

DISSIPATION IN MANY-BODY SYSTEMS: A GEOMETRIC APPROACH BASED ON INFORMATION THEORY

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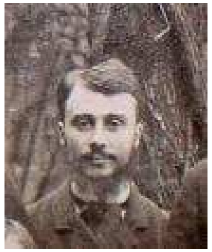
Pierre Duhem Thermodynamic Potentials $\Omega = G(E - TS) + W$

- **Duhem P., « Sur les équations générales de la Thermodynamique », Annales Scientifiques de l'Ecole Normale Supérieure, 3e série, tome VIII, p. 231, 1891**



  - "Nous avons fait de la Dynamique un cas particulier de la Thermodynamique, une Science qui embrasse dans des principes communs tous les changements d'état des corps, aussi bien les changements de lieu que les changements de qualités physiques "

- **four scientists were credited by Duhem with having carried out "the most important researches on that subject":**
  - **F. Massieu** had managed to derive Thermodynamics from a "characteristic function and its partial derivatives"
  - **J.W. Gibbs** had shown that Massieu's functions "could play the role of potentials in the determination of the states of equilibrium" in a given system.
  - **H. von Helmholtz** had put forward "similar ideas"
  - **A. von Oettingen** had given "an exposition of Thermodynamics of remarkable generality" based on general duality concept in **"Die thermodynamischen Beziehungen antithetisch entwickelt", St. Petersburg 1885**

Influence of Massieu on Poincaré

2nd edition of Poincaré Lecture on « Thermodynamics »

M. Massieu a montré que, si l'on fait choix pour variables indépendantes de v et de T ou de p et de T, il existe une fonction, d'ailleurs inconnue, de laquelle les trois fonctions des variables, p, U et S dans le premier cas, v, U et S dans le second, peuvent se déduire facilement. M. Massieu a donné à cette fonction, dont la forme dépend du choix des variables, le nom de *fonction caractéristique*.

[M. Massieu showed that, if we make choice for independent variables of v and T or of p and T, there is a function, moreover unknown, of which three functions of variables, p, U and S in the first case, v, U and S in the second, can be deducted easily. M. Massieu gave to this function, the form of which depends on the choice of variables, name of characteristic function.]

Puisque des fonctions de M. Massieu on peut déduire les autres fonctions des variables, toutes les équations de la Thermodynamique pourront s'écrire de manière à ne plus renfermer que ces fonctions et leurs dérivées; il en résultera donc, dans certains cas, une grande simplification. Nous verrons bientôt une application importante de ces fonctions.

[Because from functions of M. Massieu, we can deduct the other functions of variables, all the equations of the Thermodynamics can be written not so as to contain more than these functions and their derivatives; it will thus result from it, in certain cases, a large simplification. We shall see soon an important application of these functions.]

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Massieu Characteristic Function by Henri Poincaré

COURS DE LA FACULTÉ DES SCIENCES DE PARIS

COURS DE PHYSIQUE MATHÉMATIQUE

THERMODYNAMIQUE

PAR

H. POINCARÉ,
Membre de l'Institut.

1908

RÉDACTION DE

J. BLONDIN,
Agrégé de l'Université.

DEUXIÈME ÉDITION, REVUE ET CORRIGÉE

PARIS,
GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE
DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,
Quai des Grands-Augustins, 55.
1908

125. Fonctions caractéristiques de M. Massieu. — Le théorème de Clausius nous a conduit à l'introduction d'une nouvelle fonction de l'état d'un système : son entropie S.

Si donc nous prenons comme variables indépendantes définissant l'état du système la pression p et le volume spécifique v, nous aurons à considérer, dans les applications, trois fonctions de ces variables : la température T, l'énergie interne U et l'entropie S.

M. Massieu a montré que, si l'on fait choix pour variables indépendantes de v et de T ou de p et de T, il existe une fonction, d'ailleurs inconnue, de laquelle les trois fonctions des variables, p, U et S dans le premier cas, v, U et S dans le second, peuvent se déduire facilement. M. Massieu a donné à cette fonction, dont la forme dépend du choix des variables, le nom de *fonction caractéristique.*

Puisque des fonctions de M. Massieu on peut déduire les autres fonctions des variables, toutes les équations de la Thermodynamique pourront s'écrire de manière à ne plus renfermer que ces fonctions et leurs dérivées ; il en résultera donc, dans certains cas, une grande simplification. Nous verrons bientôt une application importante de ces fonctions.

Characteristic Function by Henri Poincaré in Probability

H. Poincaré has introduced « Characteristic Function » in Probability with <u>LAPLACE TRANSFORM</u> not with FOURIER TRANSFORM

COURS DE LA FACULTÉ DES SCIENCES DE PARIS

PUBLIÉS SOUS LES AUSPICES DE L'ASSOCIATION AMICALE DES ÉLÈVES ET ANCIENS ÉLÈVES DE LA FACULTÉ DES SCIENCES

CALCUL DES PROBABILITÉS

PAR

H. POINCARÉ

MEMBRE DE L'INSTITUT

1912

RÉDACTION DE

A. QUIQUET

ANCIEN ÉLÈVE DE L'ÉCOLE NORMALE SUPÉRIEURE

DEUXIÈME ÉDITION, REVUE ET AUGMENTÉE PAR L'AUTEUR

NOUVEAU TIRAGE 1923

PARIS

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE

55, Quai des Grands-Augustins, 55

1912

Fonctions caractéristiques. — J'appelle *fonction caractéristique* $f(\alpha)$ la valeur probable de $e^{\alpha x}$; on aura donc

$$f(\alpha)=\sum p\, e^{\alpha x},$$

si la quantité x varie d'une manière discontinue et peut prendre seulement un nombre fini de valeurs, et

$$f(\alpha)=\int \varphi(x)\, e^{\alpha x}\, dx,$$

si x varie d'une manière continue et si $\varphi(x)$ représente la loi de probabilité. Il est clair que

$$f(\alpha)=1+\frac{\alpha}{1}(x)+\frac{\alpha^2}{1.2}(x^2)+\frac{\alpha^3}{1.2.3}(x^3)+\ldots,$$

(x^p) désignant la valeur probable de x^p. On voit que $f(0)=1$.

La fonction caractéristique suffit pour définir la loi de probabilité. On a en effet par la formule de Fourier

$$f(i\alpha)=\int_{-\infty}^{+\infty}\varphi(x)\, e^{i\alpha x}\, dx,$$

$$2\pi\,\varphi(x)=\int_{-\infty}^{+\infty} f(i\alpha)\, e^{-i\alpha x}\, d\alpha.$$

Si deux quantités x et y sont *indépendantes* et si $f(\alpha)$, $f_1(\alpha)$ sont les fonctions caractéristiques correspondantes, la fonction relative à $x+y$ sera le produit $f(\alpha)\, f_1(\alpha)$. En effet, comme nous l'avons vu au paragraphe 130, la valeur probable du produit $e^{\alpha(x+y)}$ sera le produit des valeurs probables de $e^{\alpha x}$ et $e^{\alpha y}$.

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Koszul-Vinberg Characteristic Function

- **Jean-Marie Souriau has extended the Characteristic Function in Statistical Physics:**
  - looking for other kinds of invariances through co-adjoint action of a group on its momentum space
  - defining physical observables like energy, heat and momentum as pure geometrical objects.
- **In covariant Souriau model, Gibbs equilibriums states are indexed by a geometric parameter, the Geometric Temperature, with values in the Lie algebra of the dynamical Galileo/Poincaré groups, interpreted as a space-time vector (a vector valued temperature of Planck), giving to the metric tensor a null Lie derivative.**
- **Fisher Information metric appears as the opposite of the derivative of Mean "Moment map" by geometric temperature, equivalent to a Geometric Capacity or Specific Heat.**
- **We will synthetize the analogies between both Koszul and Souriau models, and will reduce their definitions to the exclusive "Inner Product" selection using symmetric bilinear "Cartan-Killing form" (introduced by Elie Cartan in 1894).**

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Elie Cartan by Henri Poincaré

RAPPORT SUR LES TRAVAUX DE M. CARTAN

fait à la Faculté des Sciences de l'Université de Paris.

PAR

H. POINCARÉ.

Si alors on dépouille la théorie mathématique de ce qui n'y apparaît que comme un accident, c'est-à-dire de sa matière, il ne restera que l'essentiel, c'est-à-dire la forme; et cette forme, qui constitue pour ainsi dire le squelette solide de la théorie, ce sera la structure du groupe.

M. CARTAN a fait faire des progrès importants à nos connaissances sur trois de ces catégories, la 1ère la 3e et la 4e. Il s'est principalement placé au point de vue le plus abstrait de la structure, de la forme pure, indépendamment de la matière, c'est-à-dire, dans l'espèce, du nombre et du choix des variables indépendantes.

Conclusions.

On voit que les problèmes traités par M. CARTAN sont parmi les plus importants, les plus abstraits et les plus généraux dont s'occupent les Mathématiques; ainsi que nous l'avons dit, la théorie des groupes est, pour ainsi dire, la Mathématique entière, dépouillée de sa matière et réduite à une forme pure. Cet extrême degré d'abstraction a sans doute rendu mon exposé un peu aride; pour faire apprécier chacun des résultats, il m'aurait fallu pour ainsi dire lui restituer la matière dont il avait été dépouillé; mais cette restitution peut se faire de mille façons différentes; et c'est cette forme unique que l'on retrouve ainsi sous une foule de vêtements divers, qui constitue le lien commun entre des théories mathématiques qu'on s'étonne souvent de trouver si voisines.

« ***A l'Exception d'Henri Poincaré qui écrivit peu avant sa mort un rapport sur les travaux d'Elie Cartan à l'occasion de la candidature de celui-ci à la Sorbonne, les mathématiciens français ne voyaient pas l'importance de l'œuvre.*** »
Paulette Libermann
La géométrie différentielle d'Elie Cartan à Charles Ehresmann et André Lichnerowicz
Géométrie au XXième siècle, HERMANN, 2005

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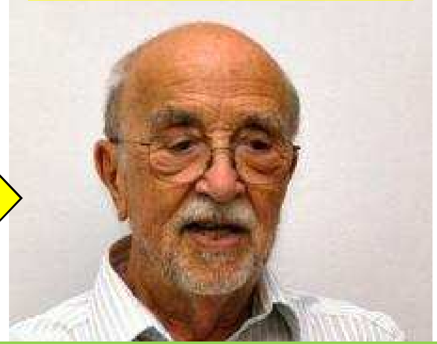
Cartanian Filiation: J.L Koszul & J.M. Souriau

Elie CARTAN

Jean-Louis Koszul

Jean-Marie Souriau



Leçons sur les invariants intégraux, Hermann, 1922

La théorie des groupes finis et continus et la géométrie différentielle (Written by J. Leray)

Koszul Forms
Koszul Characteristic Function
Koszul Hessian Metric

Souriau Moment Map
Souriau Geometric Temperature/Heat Capacity
Souriau/Fisher Metric from Symplectic Cocycle

$$\langle x,y\rangle = -B(x,\eta(y)) \text{ with } B(x,y)=Tr(ad_x ad_y)$$

where $\eta \in g$, Cartan Involution

$$\psi_\Omega(x)=\int_{\Omega^*} e^{-\langle \xi,x\rangle} d\xi \quad \forall x\in\Omega$$

$$\alpha = d\log\psi_\Omega(x)$$

$$p_x(\xi)=e^{-\langle x,\xi\rangle} \Big/ \int_{\xi\in\Omega^*} e^{-\langle x,\xi\rangle} d\xi$$

$$-E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right]=\frac{\partial^2 \log\psi_\Omega(x)}{\partial x^2}$$

$$g=D\alpha=d^2\log\psi_\Omega$$

$$f = D(\theta)(e) \text{ with } \theta \text{ cocycle associated to } G$$

$$\sigma(Z_{1,M}(\xi),Z_{2,M}(\xi))=\mu.[Z_1,Z_2]+f(Z_1,Z_2)$$

with μ Souriau Moment Map

$$f_\beta(Z_1,Z_2)=f(Z_1,Z_2)+Q.ad_{Z_1}(Z_2)$$

$$g_\beta([\beta,Z_1],Z_2)=f_\beta(Z_1,Z_2) \ ,\forall Z_1\in\mathfrak{g},\ \forall Z_2\in \mathrm{Im}(ad_\beta(.))=[\beta,.]$$

$$Temperature(\beta\in\mathfrak{g}):\beta\in Ker\ f_\beta \ ,\ C_{Heat-Capacity}=-\frac{\partial Q}{\partial\beta} \ ,\ Q\in\mathfrak{g}^*$$

$$g_\beta([\beta,Z_1],[\beta,Z_2])=f_\beta(Z_1,[\beta,Z_2])$$

$$I_{Fisher}=\frac{\partial^2\log\psi_\Omega(x)}{\partial x^2}=-\frac{\partial Q}{\partial\beta}$$

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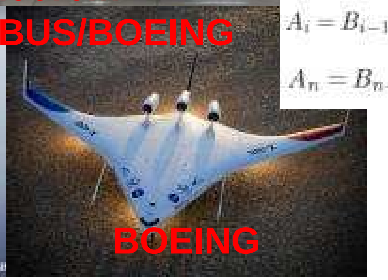
Jean-Marie Souriau and his Lie Group Thermodynamics

J.M. Souriau, "Sur la Stabilité des Avions," ONERA Publ., 62, vi+94, 1953

Engines could be positionned everywhere and a stable command could be defined

Souriau theorem revisited by AIRBUS/BOEING



J.M. Souriau, Les groupes comme universaux, Géométrie au XXième siècle, Hermann, 2005
J.M. Souriau, Grammaire de la Nature, 2007

J.M. Souriau, Calcul linéaire, P.U.F., Paris, 1964.

Multilinear Algebra

Le Verrier-Souriau Algorithm Computation of Matrix Characteristic Equation

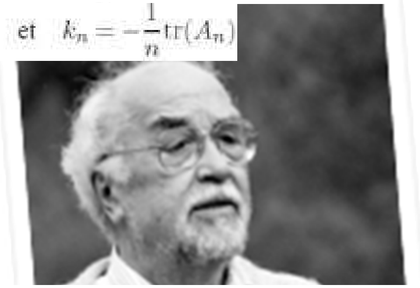
$$P(\lambda) = \det(\lambda I - A) = k_0\lambda^n + k_1\lambda^{n-1} + \cdots + k_{n-1}\lambda + k_n$$

$$Q(\lambda) = \mathrm{Adj}(\lambda I - A) = \lambda^{n-1}B_0 + \lambda^{n-2}B_1 + \cdots + \lambda B_{n-2} + B_{n-1}$$

$$k_0 = 1 \quad \text{et} \quad B_0 = I$$

$$A_i = B_{i-1}A, \qquad k_i = -\frac{1}{i}\mathrm{tr}(A_i), \qquad B_i = A_i + k_i I$$

$$A_n = B_{n-1}A \quad \text{et} \quad k_n = -\frac{1}{n}\mathrm{tr}(A_n)$$



J.M. Souriau, Structure des systèmes dynamiques, Dunod, Paris, 1970

Symplectic Geometry Structure of Classical & Quantum Mechanics

- **Moment Map**
- **Geometric Noether Theorem**
- **General Barycentric Theorem**
- **Mass = Symplectic cohomology of the action of the Galilean group (not for Poincaré Group in Relativity)**

« La masse totale d'un système dynamique isolé est la classe de cohomologie du défaut d'équivariance de l'application moment »

J.M. Souriau, Définition covariante des équilibres thermodynamiques, Supp. Nuov. Cimento,1,4, p.203-216, 1966
J.M. Souriau, Thermodynamique et géométrie. In diff. Geo. methods in mathematical physics, II, vol. 676 of Lecture Notes in Math., pages 369-397. Springer, 1978

Lie Group Thermodynamics
(Gibbs Equilibrium is not covariant by Gallileo/Poincaré Groups)

- **Geometric Temperature (Vector in Lie Algebra of Dynamical Group) & Geometric Entropy**
- **FISHER METRIC DEFINED THROUGH SYMPLECTIC COCYCLE OF DYNAMICAL GROUP**
- **FISHER METRIC = GEOMETRIC HEAT CAPACITY**

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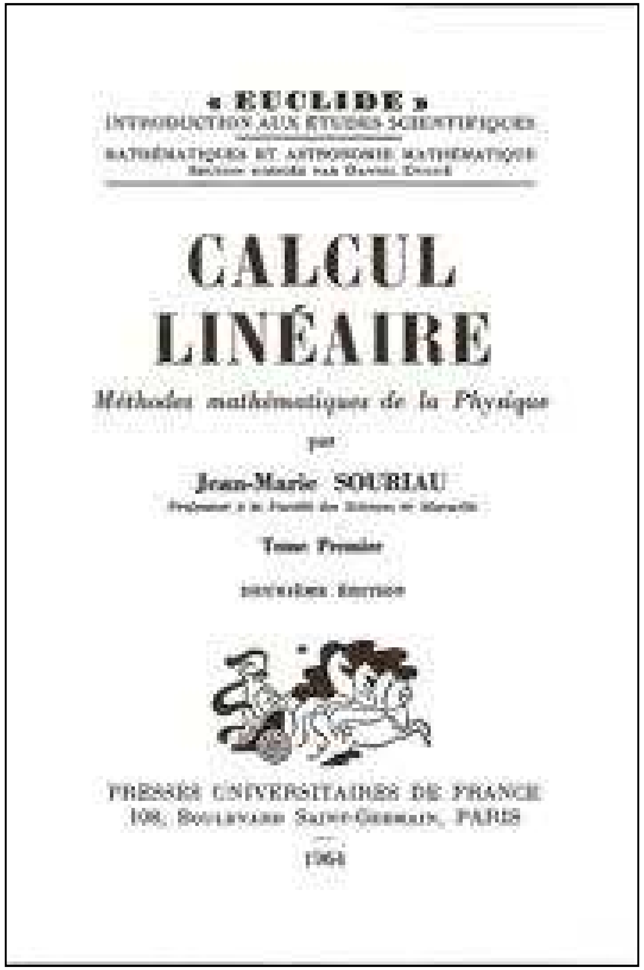
Souriau Books


« EUCLIDE »
INTRODUCTION AUX ÉTUDES SCIENTIFIQUES
MATHÉMATIQUES ET ASTRONOMIE MATHÉMATIQUE

CALCUL LINÉAIRE

Méthodes mathématiques de la Physique

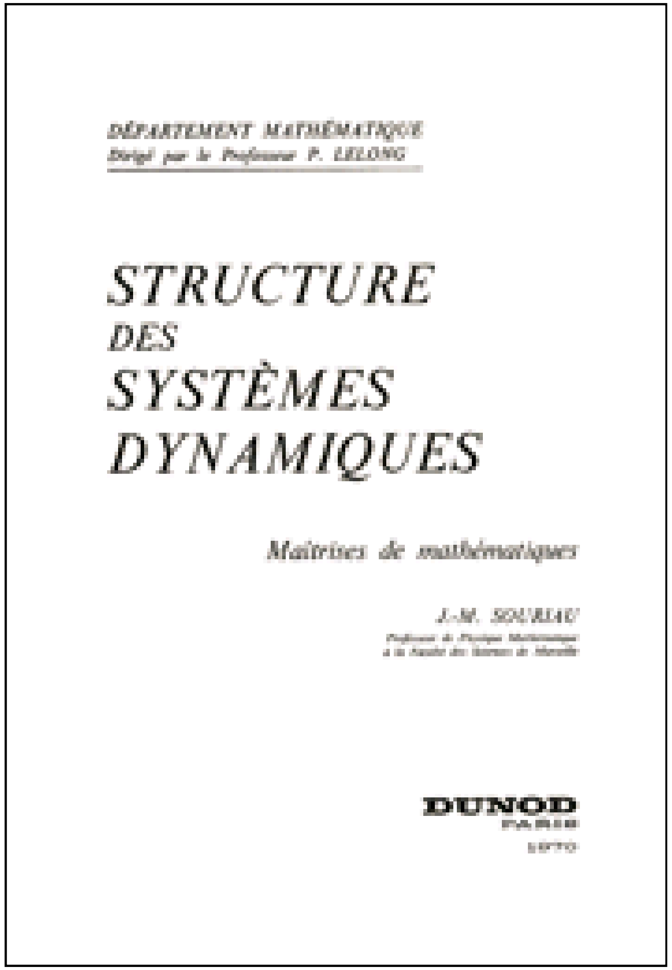
par

Jean-Marie SOURIAU

Tome Premier

DEUXIÈME ÉDITION

PRESSES UNIVERSITAIRES DE FRANCE
108, Boulevard Saint-Germain, PARIS
1964


DÉPARTEMENT MATHÉMATIQUE
Dirigé par le Professeur P. LELONG

STRUCTURE DES SYSTÈMES DYNAMIQUES

Maîtrises de mathématiques

J.-M. SOURIAU

DUNOD
PARIS
1970

http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm

Chapter 4 on « Statistical Mechanics »
http://www.jmsouriau.com/Publications/JMSouriau-SSD-Ch4.pdf

Souriau Model for Gaussian Law

J.M. Souriau, Structure des systèmes dynamiques, Chapitre 4 « Mécanique Statistique »

Exemple : (*loi normale*) :

Prenons le cas $V = R^n$, λ = mesure de Lebesgue, $\Psi(x) \equiv \begin{pmatrix} x \\ x \otimes x \end{pmatrix}$; un élément Z du dual de E peut se définir par la formule

$$Z(\Psi(x)) \equiv \overline{a}.x + \tfrac{1}{2}\overline{x}.H.x$$

[$a \in R^n$; H = matrice symétrique]. On vérifie que la convergence de l'intégrale I_0 a lieu si la matrice H est positive ([1]) ; dans ce cas la loi de Gibbs s'appelle *loi normale de Gauss* ; on calcule facilement I_0 en faisant le changement de variable $x^* = H^{1/2}\,x + H^{-1/2}\,a$ ([2]) ; il vient

$$z = \tfrac{1}{2}\left[\overline{a}.H^{-1}.a - \log(\text{dét}(H)) + n\log(2\pi)\right]$$

alors la convergence de I_1 a lieu également ; on peut donc calculer M, qui est défini par les moments du premier et du second ordre de la loi (16.196) ; le calcul montre que le moment du premier ordre est égal à $-H^{-1}.a$ et que les composantes du tenseur *variance* (16.196) sont égales aux éléments de la matrice H^{-1} ; le moment du second ordre s'en déduit immédiatement.

La formule (16.200♡) donne l'*entropie* :

$$s = \frac{n}{2}\log(2\pi e) - \frac{1}{2}\log(\text{dét}(H))$$;.

([1]) Voir *Calcul linéaire*, tome II.
([2]) C'est-à-dire en recherchant l'*image* de la loi par l'application $x \mapsto x^*$.

Exemple

Soit x une variable aléatoire. Posons $F(x) \equiv \begin{pmatrix} x \\ x \otimes x \end{pmatrix}$; on vérifie que l'ensemble Q des $\begin{pmatrix} {}^1M \\ {}^2M \end{pmatrix}$, où ${}^1M$ et ${}^2M$ sont des tenseurs contravariants de E, respectivement de degré 1 et 2, tels que ([1])

$$1 + 2\,{}^1M(C) + {}^2M(C)(C) \geqslant 0 \qquad \forall C \in E^*$$

est un ensemble fermé convexe contenant val (F) ; on a donc

$$\text{moyenne } F(x) \in Q,$$

ce qui s'écrit

$$1 + 2\,{}^1M(C) + {}^2M(C)(C) \geqslant 0 \qquad \forall C \in E^*$$

${}^1M$ et ${}^2M$ désignant les *moments* d'ordre 1 et 2 de x ; on peut aussi écrire cette propriété sous la forme

$$\Phi(C)(C) \geqslant 0 \qquad \forall C \in E^*,$$

Φ étant le tenseur ${}^2M - {}^1M \otimes {}^1M$; Φ s'appelle tenseur *variance* de x.

Souriau « Geometric Temperature » idea come from his book « Calcul Linéaire » (chap. on «Multilinear Algebra »

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Souriau Mechanical Statistics

J.M. Souriau, Structure des systèmes dynamiques, Chapitre 4 « Mécanique Statistique »

◇

$$I_0 = \int_U e^{-\mu Z}\, \varphi(dx)$$

$$I_1 = \int_U \mu\, e^{-\mu Z}\, \varphi(dx)$$

$$I_2 = \int_U \mu \otimes \mu\, e^{-\mu Z}\, \varphi(dx)$$

convergent, et que les applications $Z \mapsto I_0$, $Z \mapsto I_1$ soient une fois différentiables sous le signe $\int$ ([1]).

Si $Z \in \Omega$, nous poserons

♡ $$z \equiv \log(I_0)$$

♤ $$M \equiv \frac{I_1}{I_0}$$

2) Si $Z \in \Omega$, il existe une loi de probabilité ζ, définie par la densité

♧ $$e^{-[z+\mu Z]}\,\varphi \qquad \text{(loi de Gibbs)}$$

elle vérifie ([2])

☾ $$\text{moyenne } \mu = M$$

son entropie est

♂ $$s = z + M.Z$$

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Covariant Souriau Statistical Mechanics

MÉCANIQUE STATISTIQUE COVARIANTE

Le groupe des translations dans le temps (7.9) est un sous-groupe du groupe de Galilée ; mais *ce n'est pas un sous-groupe invariant*, ainsi que le montre un calcul trivial. Si un système dynamique est *conservatif* dans un repère d'inertie, il en résulte qu'il peut *ne plus être conservatif dans un autre*. La formulation (17.24) du principe de Gibbs doit donc être élargie, pour devenir compatible avec la relativité galiléenne.

Nous proposons donc le principe suivant :

(17.77) Si un système dynamique est invariant par un sous-groupe de Lie G' du groupe de Galilée, les équilibres naturels du système constituent l'ensemble de Gibbs du groupe dynamique G'.

Soit $\mathcal{G}'$ l'algèbre de Lie G' ; on sait que $\mathcal{G}'$ est une sous-algèbre de Lie de celle de G, notée $\mathcal{G}$; un équilibre du système sera caractérisé par un élément Z de $\mathcal{G}'$, donc de $\mathcal{G}$; on pourra écrire

(17.78) $$Z = \begin{bmatrix} j(\omega) & \beta & \gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}$$

en utilisant les notations (13.4) ; Z parcourt l'ensemble Ω défini en (16.219) ; à chaque valeur de Z est associé un élément M du dual $\mathcal{G}'^*$ de $\mathcal{G}'$, valeur moyenne du moment μ ; on peut appliquer les formules (16.219), (16.220), qui généralisent les relations thermodynamiques (17.26), (17.27), (17.28).

(17.79) On voit que c'est Z (17.78) qui généralise la « température » ; le théorème

d'isothermie (17.32) s'étend immédiatement : l'équilibre d'un système composé de plusieurs parties sans interactions s'obtient en attribuant à chaque composante un équilibre *correspondant à la même valeur de Z* ; l'entropie s, le potentiel de Planck z et le moment moyen M sont *additifs*. W

J.M. Souriau, Structure des systèmes dynamiques, Chapitre 4 « Mécanique Statistique »

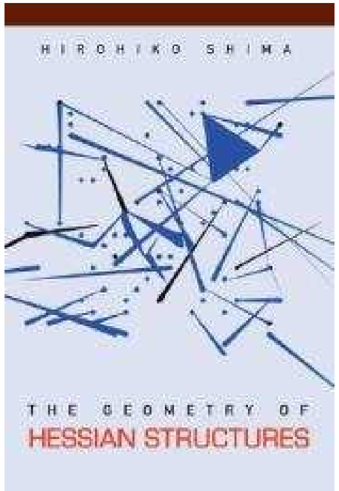
Classical Gibbs Equilibrium is not covariant according to Dynamic Group of Mechanics (Gallileo Group and Poincaré Group) !!!

THALES

Hessian Geometry by J.L. Koszul

- Hirohiko Shima Book, « Geometry of Hessian Structures », world Scientific Publishing 2007, dedicated to Jean-Louis Koszul
- Hirohiko Shima Keynote Talk at GSI'13
  - http://www.see.asso.fr/file/5104/download/9914
- Prof. M. Boyom tutorial :
  - http://repmus.ircam.fr/_media/brillouin/ressources/une-source-de-nouveaux-invariants-de-la-geometrie-de-l-information.pdf


Jean-Louis Koszul



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J.L. Koszul, « Domaines bornées homogènes et orbites de groupes de transformations affines », Bull. Soc. Math. France 89, pp. 515-533., 1961
J.L. Koszul, « Ouverts convexes homogènes des espaces affines », Math. Z. 79, pp. 254-259., 1962
J.L. Koszul, « Variétés localement plates et convexité », Osaka J. Maht. 2, pp. 285-290., 1965
J.L. Koszul, « Déformations des variétés localement plates », .Ann Inst Fourier, 18 , 103-114., 1968

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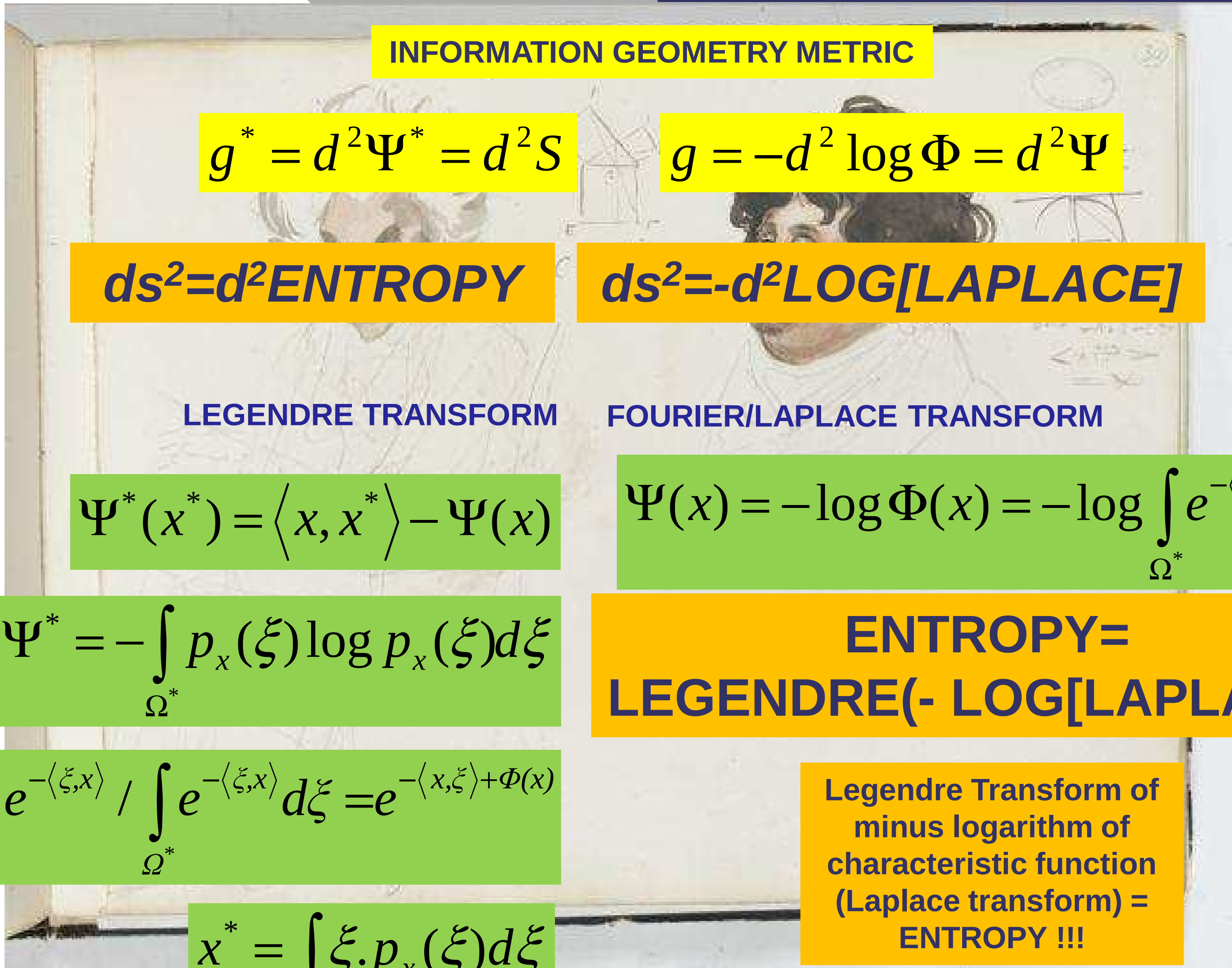
Koszul Information Geometry

THALES

Projective Legendre Duality and Koszul Characteristic Function



INFORMATION GEOMETRY METRIC

$$g^* = d^2\Psi^* = d^2 S$$

$$g = -d^2 \log \Phi = d^2 \Psi$$

ds^2=d^2ENTROPY

ds^2=-d^2LOG[LAPLACE]

LEGENDRE TRANSFORM

$$\Psi^*(x^*) = \left\langle x, x^* \right\rangle - \Psi(x)$$

$$\Psi^* = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi$$

$$p_x(\xi) = e^{-\langle \xi, x \rangle} / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi = e^{-\langle x, \xi \rangle + \Phi(x)}$$

$$x^* = \int_{\Omega^*} \xi . p_x(\xi) d\xi$$

FOURIER/LAPLACE TRANSFORM

$$\Psi(x) = -\log \Phi(x) = -\log \int_{\Omega^*} e^{-\langle x, y \rangle} dy$$

ENTROPY=
LEGENDRE(- LOG[LAPLACE])

Legendre Transform of minus logarithm of characteristic function (Laplace transform) = ENTROPY !!!

THALES

Koszul-Vinberg Characteristic Function/Metric of convex cone

- J.L. Koszul and E. Vinberg have introduced an affinely invariant Hessian metric on a sharp convex cone through its characteristic function.
- Ω is a sharp open convex cone in a vector space E of finite dimension on R (a convex cone is sharp if it does not contain any full straight line).
- Ω^* is the dual cone of Ω and is a sharp open convex cone.
- Let $d\xi$ the Lebesgue measure on E^* dual space of E, the following integral:

$$\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega$$

is called the **Koszul-Vinberg characteristic function**

Koszul-Vinberg Characteristic Function/Metric of convex cone

- **Koszul-Vinberg Metric :** $g = d^2 \log \psi_\Omega$

$$d^2 \log \psi(x) = d^2 \left[\log \int \psi_u du \right] = \frac{\int \psi_u d^2 \log \psi_u du}{\int \psi_u du} + \frac{1}{2} \frac{\iint \psi_u \psi_v (d \log \psi_u - d \log \psi_v)^2 dudv}{\iint \psi_u \psi_v dudv}$$

- **We can define a diffeomorphism by:** $x^* = -\alpha_x = -d \log \psi_\Omega(x)$

with $\langle df(x), u \rangle = D_u f(x) = \left. \frac{d}{dt} \right|_{t=0} f(x + tu)$

- **When the cone** Ω **is symmetric, the map** $x^* = -\alpha_x$ **is a bijection and an isometry with a unique fixed point (the manifold is a Riemannian Symmetric Space given by this isometry):**

$(x^*)^* = x$, $\langle x, x^* \rangle = n$ **and** $\psi_\Omega(x) \psi_{\Omega^*}(x^*) = cste$

- x^* **is characterized by** $x^* = \arg\min \{ \psi(y) / y \in \Omega^*, \langle x, y \rangle = n \}$

- x^* **is the center of gravity of the cross section** $\{ y \in \Omega^*, \langle x, y \rangle = n \}$ **of** Ω^* **:**

$$x^* = \int_{\Omega^*} \xi . e^{-\langle \xi, x \rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

Koszul Entropy via Legendre Transform

- **we can deduce "Koszul Entropy" defined as Legendre Transform of minus logarithm of Koszul-Vinberg characteristic function** $\Phi(x) = -\log \psi_\Omega(x)$ **:**

$$\Phi^*(x^*) = \langle x, x^* \rangle - \Phi(x)$$ **with** $x^* = D_x\Phi$ **and** $x = D_{x^*}\Phi^*$ **where**

- **Demonstration:** **we set** $\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega$

Using $x^* = \int_{\Omega^*} \xi . e^{-\langle \xi, x \rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$

and $\langle -x^*, h \rangle = d_h \log \psi_\Omega(x) = -\int_{\Omega^*} \langle \xi, h \rangle e^{-\langle \xi, x \rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$

we can write: $-\langle x^*, x \rangle = \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$

and

$$\Phi^*(x^*) = -\int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi + \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

$$\Phi^*(x^*) = \left[\left(\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \right) . \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \right] \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

Koszul-Vinberg Characteristic Function Legendre Transform

$$\Phi^*(x^*) = \left\langle x, x^* \right\rangle - \Phi(x) = -\int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi + \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

$$\Phi^*(x^*) = \left[\left(\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \right) . \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \right] / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

$$\Phi^*(x^*) = \left[\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right]$$

$$\Phi^*(x^*) = \left[\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi . \left(\int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right) - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right] \text{ with } \int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi = 1$$

$$\Phi^*(x^*) = \left[-\int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} . \log \left(\frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} \right) d\xi \right]$$

Koszul Entropy via Legendre Transform

- We can then consider this Legendre transform as an entropy, that we could named "**Koszul Entropy**":

$$\Phi^* = -\int_{\Omega^*} \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} \log \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} d\xi = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi$$

With

$$p_x(\xi) = e^{-\langle \xi, x\rangle} / \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi = e^{-\langle x, \xi\rangle - \log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} = e^{-\langle x, \xi\rangle + \Phi(x)}$$

and

$$x^* = D_x \Phi = \int_{\Omega^*} \xi . p_x(\xi) d\xi = \int_{\Omega^*} \xi . e^{-\langle x, \xi\rangle + \Phi(x)} d\xi = \int_{\Omega^*} \xi . e^{-\Phi^*(\xi)} d\xi$$

$$\Phi(x) = -\log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi = -\log \int_{\Omega^*} e^{-[\Phi^*(\xi) + \Phi(x)]} d\xi$$

$$\Phi(x) = \Phi(x) - \log \int_{\Omega^*} e^{-\Phi^*(\xi)} d\xi \Rightarrow \int_{\Omega^*} e^{-\Phi^*(\xi)} d\xi = 1$$

$$\text{Jensen Ineq.}: \Phi^* \text{ conv.} \Rightarrow \Phi^*(E[\xi]) \leq E[\Phi^*(\xi)]$$

$$\text{Legendre Transform}: \Phi^*(x^*) \geq \langle x, x^* \rangle - \Phi(x)$$

$$\Rightarrow \Phi^*(x^*) \geq \int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = E[\Phi^*(\xi)]$$

$$\log p_x(\xi) = \log e^{-\langle x, \xi\rangle + \Phi(x)} = \log e^{-\Phi^*(\xi)} = -\Phi^*(\xi)$$

$$\Rightarrow -\int_{\Omega^*} \log p_x(\xi) p_x(\xi) d\xi = \int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = \Phi^*(x^*)$$

$$\text{if and only if } \int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = \Phi^*\left(\int_{\Omega^*} \xi . p_x(\xi) d\xi \right)$$

$$\text{or } E[\Phi^*(\xi)] = \Phi^*(E[\xi])$$

THALES

Barycenter of Koszul Entropy = Koszul Entropy of Barycenter

$$E\left[\Phi^{*}(\xi)\right]=\Phi^{*}\left(E[\xi]\right)$$

$$\int_{\Omega^*}\Phi^{*}(\xi)p_x(\xi)d\xi=\Phi^{*}\left(\int_{\Omega^*}\xi.p_x(\xi)d\xi\right)$$

$$p_x(\xi)=e^{-\langle\xi,x\rangle}/\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi=e^{-\langle x,\xi\rangle-\log\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi}=e^{-\langle x,\xi\rangle+\Phi(x)}$$

$$x^{*}=D_x\Phi=\int_{\Omega^*}\xi.p_x(\xi)d\xi=\int_{\Omega^*}\xi.e^{-\langle x,\xi\rangle+\Phi(x)}d\xi=\int_{\Omega^*}\xi.e^{-\Phi^{*}(\xi)}d\xi$$

$$\Phi^{*}(x^{*})=\underset{x}{Sup}\left[\langle x,x^{*}\rangle-\Phi(x)\right]$$

THALES

Koszul metric & Fisher Metric

- **To make the link with Fisher metric given by matrix $I(x)$, we can observe that the second derivative of $\log p_x(\xi)$ is given by:**

$$\log p_x(\xi) = -\Phi^*(\xi) = \Phi(x) - \langle x, \xi \rangle$$

$$\frac{\partial^2 \log p_x(\xi)}{\partial x^2} = \frac{\partial^2 [\Phi(x) - \langle x, \xi \rangle]}{\partial x^2} = \frac{\partial^2 \Phi(x)}{\partial x^2}$$

$$\Rightarrow I(x) = -E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right] = -\frac{\partial^2 \Phi(x)}{\partial x^2} = \frac{\partial^2 \log \psi_\Omega(x)}{\partial x^2}$$

- **We could then deduce the close interrelation between Fisher metric and hessian of Koszul-Vinberg characteristic logarithm.**

$$I(x) = -E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right] = \frac{\partial^2 \log \psi_\Omega(x)}{\partial x^2}$$

FISHER METRIC (Information Geometry) =
KOSZUL HESSIAN METRIC (Hessian Geometry)

Koszul Metric and Fisher Metric as Variance

- We can also observed that the Fisher metric or hessian of KVCF logarithm is related to the variance of ξ :

$$\log \Psi_{\Omega}(x) = \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \Rightarrow \frac{\partial \log \Psi_{\Omega}(x)}{\partial x} = -\frac{1}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} \int_{\Omega^*} \xi . e^{-\langle \xi, x \rangle} d\xi$$

$$\frac{\partial^2 \log \Psi_{\Omega}(x)}{\partial x^2} = -\frac{1}{\left(\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \right)^2} \left[- \int_{\Omega^*} \xi^2 . e^{-\langle \xi, x \rangle} d\xi . \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi + \left(\int_{\Omega^*} \xi . e^{-\langle \xi, x \rangle} d\xi \right)^2 \right]$$

$$\frac{\partial^2 \log \Psi_{\Omega}(x)}{\partial x^2} = \int_{\Omega^*} \xi^2 . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi - \left(\int_{\Omega^*} \xi . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right)^2 = \int_{\Omega^*} \xi^2 . p_x(\xi) d\xi - \left(\int_{\Omega^*} \xi . p_x(\xi) d\xi \right)^2$$

$$I(x) = -E_{\xi}\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2} \right] = \frac{\partial^2 \log \psi_{\Omega}(x)}{\partial x^2} = E_{\xi}\left[\xi^2\right] - E_{\xi}[\xi]^2 = Var(\xi)$$

THALES

New Definition of Maximum Entropy Density

- How to replace x by mean value of ξ , $\bar{\xi}(= x^*)$ in :

$$p_x(\xi) = \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} \quad \text{with} \quad \bar{\xi} = \int_{\Omega^*} \xi . p_x(\xi) d\xi$$

- Legendre Transform will do this inversion by inversing $\bar{\xi} = \frac{d\Phi(x)}{dx}$
- We then observe that Koszul Entropy provides density of Maximum Entropy with this general definition of density:

$$p_{\bar{\xi}}(\xi) = \frac{e^{-\langle \xi, \Theta^{-1}(\bar{\xi})\rangle}}{\int_{\Omega^*} e^{-\langle \xi, \Theta^{-1}(\bar{\xi})\rangle} d\xi} \quad \text{with} \quad x = \Theta^{-1}(\bar{\xi}) \quad \text{and} \quad \bar{\xi} = \Theta(x) = \frac{d\Phi(x)}{dx}$$

where $\bar{\xi} = \int_{\Omega^*} \xi . p_{\bar{\xi}}(\xi) d\xi$ and $\Phi(x) = -\log \int_{\Omega^*} e^{-\langle x, \xi\rangle} d\xi$

Cartan-Killing Form and Invariant Inner Product

- It is not possible to define an ad(g)-invariant inner product for any two elements of a Lie Algebra, but a symmetric bilinear form, called "**Cartan-Killing form**", could be introduced (Elie Cartan PhD 1894)
- This form is defined according to the adjoint endomorphism ad_x of g that is defined for every element x of g with the help of the Lie bracket: $ad_x(y) = [x, y]$
- The trace of the composition of two such endomorphisms defines a bilinear form, the **Cartan-Killing form**:

$$B(x, y) = Tr(ad_x ad_y)$$

- The Cartan-Killing form is symmetric: $B(x, y) = B(y, x)$
- and has the associativity property: $B([x, y], z) = B(x, [y, z])$
- given by:

$$B([x, y], z) = Tr(ad_{[x,y]} ad_z) = Tr([ad_x, ad_y] ad_z)$$

$$B([x, y], z) = Tr(ad_x [ad_y, ad_z]) = B(x, [y, z])$$

THALES

Cartan-Killing Form and Invariant Inner Product

- **Elie Cartan has proved that if g is a simple Lie algebra (the Killing form is non-degenerate) then any invariant symmetric bilinear form on g is a scalar multiple of the Cartan-Killing form.**
- **The Cartan-Killing form is invariant under automorphisms $\sigma \in Aut(g)$ of the algebra g :**

$$B(\sigma(x),\sigma(y)) = B(x,y)$$

- **To prove this invariance, we have to consider:**

$$\begin{cases} \sigma[x,y] = [\sigma(x),\sigma(y)] \\ z = \sigma(y) \end{cases} \Rightarrow \sigma[x,\sigma^{-1}(z)] = [\sigma(x),z]$$

$$\text{rewritten} \quad ad_{\sigma(x)} = \sigma \circ ad_x \circ \sigma^{-1}$$

Then

$$B(\sigma(x),\sigma(y)) = Tr(ad_{\sigma(x)} ad_{\sigma(y)}) = Tr(\sigma \circ ad_x ad_y \circ \sigma^{-1})$$

$$B(\sigma(x),\sigma(y)) = Tr(ad_x ad_y) = B(x,y)$$

THALES

Cartan-Killing Form and Invariant Inner Product

A natural G-invariant inner product could be introduced by Cartan-Killing form:

- **Cartan Generating Inner Product: The following Inner product defined by Cartan-Killing form is invariant by automorphisms of the algebra**

  $$\langle x, y \rangle = -B\big(x, \theta(y)\big)$$

  where $\theta \in g$ **is a Cartan involution (An involution on** g **is a Lie algebra automorphism** θ **of** g **whose square is equal to the identity).**

From Cartan-Killing Form to Koszul Information Metric

$$B(x,y)=Tr\left(ad_x ad_y\right)$$

Cartan – Killing Form

$$\langle x,y\rangle=-B(x,\theta(y))$$

with $\theta\in g$, Cartan Involution

Koszul Characteristic Function

$$\Phi(x)=-\log\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi \quad \forall x\in\Omega$$

Koszul Entropy

$$\Phi^*(x^*)=\langle x,x^*\rangle-\Phi(x)$$

$$\Phi^*(x^*)=-\int_{\Omega^*}p_x(\xi)\log p_x(\xi)d\xi$$

with $x^*=\int_{\Omega^*}\xi.p_x(\xi)d\xi$

Koszul Density

$$p_x(\xi)=\frac{e^{-\langle\xi,x\rangle}}{\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi}$$

Koszul Metric

$$I(x)=-E_\xi\left[\frac{\partial^2\log p_x(\xi)}{\partial x^2}\right]$$

$$I(x)==-\frac{\partial^2\Phi(x)}{\partial x^2}=\frac{\partial^2\log\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi}{\partial x^2}$$

Koszul Density: Application for SPD matrices

- **We can then named this new density as "Koszul Density":**

$$p_x(\xi) = e^{-\langle \xi, x\rangle} / \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi = e^{-\langle x,\xi\rangle - \log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} = e^{-\langle x,\xi\rangle + \Phi(x)}$$

With $$x^* = D_x\Phi = \int_{\Omega^*} \xi . p_x(\xi) d\xi = \int_{\Omega^*} \xi . e^{-\langle x,\xi\rangle + \Phi(x)} d\xi = \int_{\Omega^*} \xi . e^{-\Phi^*(\xi)} d\xi$$

$$\begin{cases} \langle x, y\rangle = Tr(xy), \forall x, y \in Sym_n(R) \\ \psi_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi \underset{\substack{\langle x,y\rangle = Tr(xy) \\ \Omega^* = \Omega \text{ self-dual}}}{=} \det x^{-\frac{n+1}{2}} \psi(I_n) \\ x^* = \bar{\xi} = -d \log \psi_\Omega = \frac{n+1}{2} d \log \det x = \frac{n+1}{2} x^{-1} \end{cases}$$

$$p_x(\xi) = e^{-Tr(x\xi) + \frac{n+1}{2}\log\det x} = \left[\det\left(\alpha\bar{\xi}^{-1}\right)\right]^{\alpha} e^{-Tr\left(\alpha\bar{\xi}^{-1}\xi\right)} \text{ with } \bar{\xi} = \int_{\Omega^*} \xi . p_x(\xi) . d\xi$$

Relation of Koszul density with Maximum Entropy Principle

- The density from Maximum Entropy Principle is given by:

$$\underset{p_x(.)}{Max}\left[-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi\right] \text{ such } \begin{cases}\int_{\Omega^*} p_x(\xi)d\xi = 1 \\ \int_{\Omega^*} \xi.p_x(\xi)d\xi = x^* \end{cases}$$

- If we take $q_x(\xi) = e^{-\langle \xi,x\rangle} / \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi = e^{-\langle x,\xi\rangle - \log \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi}$ such that:

$$\begin{cases}\int_{\Omega^*} q_x(\xi).d\xi = \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi / \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi = 1 \\ \log q_x(\xi) = \log e^{-\langle x,\xi\rangle - \log \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi} = -\langle x,\xi\rangle - \log \int_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi \end{cases}$$

- Then by using the fact that $\log x \geq \left(1 - x^{-1}\right)$ with equality if and only if $x = 1$, we find the following:

$$-\int_{\Omega^*} p_x(\xi)\log\frac{p_x(\xi)}{q_x(\xi)}d\xi \leq -\int_{\Omega^*} p_x(\xi)\left(1 - \frac{q_x(\xi)}{p_x(\xi)}\right)d\xi$$

THALES

Relation of Koszul density with Maximum Entropy Principle

- **We can then observe that:**

$$\int_{\Omega^*} p_x(\xi)\left(1-\frac{q_x(\xi)}{p_x(\xi)}\right)d\xi = \int_{\Omega^*} p_x(\xi)d\xi - \int_{\Omega^*} q_x(\xi)d\xi = 0$$

because $\int_{\Omega^*} p_x(\xi)d\xi = \int_{\Omega^*} q_x(\xi)d\xi = 1$

- **We can then deduce that:**

$$-\int_{\Omega^*} p_x(\xi)\log\frac{p_x(\xi)}{q_x(\xi)}d\xi \le 0 \Rightarrow -\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \le -\int_{\Omega^*} p_x(\xi)\log q_x(\xi)d\xi$$

- **If we develop the last inequality, using expression of** $q_x(\xi)$**:**

$$-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \le -\int_{\Omega^*} p_x(\xi)\left[-\langle x,\xi\rangle - \log\int_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi\right]d\xi$$

$$-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \le \left\langle x, \int_{\Omega^*} \xi.p_x(\xi)d\xi\right\rangle + \log\int_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi$$

$$-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \le \langle x, x^*\rangle - \Phi(x)$$

$$-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \le \Phi^*(x^*)$$

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Souriau Lie Group Thermodynamics

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Covariant Definition of Gibbs Equilibrium by Souriau

Covariant Definition of Thermodynamic Equilibriums

- **Jean-Marie Souriau , student of Elie Cartan at ENS Ulm in 1946, has**
  - given a **covariant definition of thermodynamic equilibriums**
  - formulated statistical mechanics and thermodynamics in the framework of **Symplectic Geometry**

by use of symplectic moments and distribution-tensor concepts, giving a geometric status for:

  - Temperature
  - Heat
  - Entropy

- **This work has been extended by C. Vallée & G. de Saxcé, P. Iglésias and F. Dubois.**

Covariant Definition of Thermodynamic Equilibriums

- **The first general definition of the "moment map" (constant of the motion for dynamical systems) was introduced by Jean-Marie Souriau during 1970's**
  - with geometric generalization such earlier notions as the Hamiltonian and the invariant theorem of Emmy Noether describing the connection between symmetries and invariants (it is the moment map for a one-dimensional Lie group of symmetries).
- **In symplectic geometry the analog of Noether's theorem is the statement that the moment map of a Hamiltonian action which preserves a given time evolution is itself conserved by this time evolution.**
- **The conservation of the moment of a Hamilotnian action was called by Souriau the "Symplectic or Geometric Noether theorem"**
  - considering phases space as symplectic manifold, cotangent fiber of configuration space with canonical symplectic form, if Hamiltonian has Lie algebra, **moment map is constant along system integral curves**.
  - **Noether theorem is obtained by considering independently each component of moment map**

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Souriau Covariant Model

- Let M be a differentiable manifold with a continuous positive density $d\omega$ and let E a finite vector space and $U(\xi)$ a continuous function defined on M with values in E. A continuous positive function $p(\xi)$ solution of this problem with respect to calculus of variations:

$$\underset{p(\xi)}{ArgMin}\left[s = -\int_M p(\xi)\log p(\xi) d\omega \right] \text{ such that } \begin{cases} \int_M p(\xi) d\omega = 1 \\ \int_M U(\xi) p(\xi) d\omega = Q \end{cases}$$

- is given by:

$$p(\xi) = e^{\Phi(\beta) - \beta.U(\xi)} \text{ and } Q = \frac{\int_M U(\xi) e^{-\beta.U(\xi)} d\omega}{\int_M e^{-\beta.U(\xi)} d\omega}$$

and $\Phi(\beta) = -\log \int_M e^{-\beta.U(\xi)} d\omega$

- Entropy $s = -\int_M p(\xi)\log p(\xi) d\omega$ can be stationary only if there exist a scalar Φ and an element β belonging to the dual of E.
- Entropy appears naturally as Legendre transform of Φ :

$$s(Q) = \beta.Q - \Phi(\beta)$$

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Souriau Covariant Model

- This value $s(Q) = \beta.Q - \Phi(\beta)$ is a strict minimum of s, and the equation:

$$Q = \frac{\int_M U(\xi) e^{-\beta.U(\xi)} d\omega}{\int_M e^{-\beta.U(\xi)} d\omega}$$

has a maximum of one solution for each value of Q.

- The function $\Phi(\beta)$ is differentiable and we can write $d\Phi = d\beta.Q$ and identifying *E* with its dual: $Q = \frac{\partial \Phi}{\partial \beta}$
- Uniform convergence of $\int_M U(\xi) \otimes U(\xi) e^{-\beta.U(\xi)} d\omega$ proves that $-\frac{\partial^2 \Phi}{\partial \beta^2} > 0$ and that $-\Phi(\beta)$ is convex.
- Then, $Q(\beta)$ and $\beta(Q)$ are mutually inverse and differentiable, where $ds = \beta.dQ$.
- Identifying E with its bidual: $\beta = \frac{\partial s}{\partial Q}$

Souriau-Gibbs Canonical Ensemble

- **In statistical mechanics, a canonical ensemble is the statistical ensemble that is used to represent the possible states of a mechanical system that is being maintained in thermodynamic equilibrium.**
- **Souriau has defined this Gibbs canonical ensemble on Symplectic manifold *M* for a Lie group action on *M***
- **The seminal idea of Lagrange was to consider that a statistical state is simply a probability measure on the manifold of motions**
- **In Jean-Marie Souriau approach, one movement of a dynamical system (classical state) is a point on manifold of movements.**
- **For statistical mechanics, the movement variable is replaced by a random variable where a statistical state is probability law on this manifold.**

Souriau-Gibbs Canonical Ensemble

- **Symplectic manifolds have a completely continuous measure, invariant by diffeomorphisms: the Liouville measure** λ
- **All statistical states will be the product of Liouville measure by the scalar function given by the generalized partition function** $e^{\Phi-\beta.U}$ **defined by the generalized energy** U **(the moment that is defined in dual of Lie Algebra of this dynamical group) and the geometric temperature** β **, where** Φ **is a normalizing constant such the mass of probability is equal to 1,** $\Phi = -\log \int_M e^{-\beta.U} d\omega$
- **Jean-Marie Souriau generalizes the Gibbs equilibrium state to all Symplectic manifolds that have a dynamical group.**
- **To ensure that all integrals could converge, the canonical Gibbs ensemble is the largest open proper subset (in Lie algebra) where these integrals are convergent. This canonical Gibbs ensemble is convex.**
  - the mean value of the energy $Q = \frac{\partial \Phi}{\partial \beta}$
  - a generalization of heat capacity $K = -\frac{\partial Q}{\partial \beta}$
  - Entropy by Legendre transform $s = \beta.Q - \Phi$

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Souriau Lie Group Thermodynamic

- **For the group of time translation, this is the classical thermodynamic**
- **Souriau has observed that if we apply this theory for non-commutative group (Galileo or Poincaré groups):**
  - the symmetry has been broken
  - Classical Gibbs equilibrium states are no longer invariant by this group
- **This symmetry breaking provides new equations, discovered by Jean-Marie Souriau.**
- **For each temperature β , Jean-Marie Souriau has introduced a tensor f_β , equal to the sum of cocycle f and Heat coboundary (with [.,.] Lie bracket):**

$$f_\beta(Z_1,Z_2)= f(Z_1,Z_2)+Q.ad_{Z_1}(Z_2) \quad \text{with} \quad ad_{Z_1}(Z_2)=[Z_1,Z_2]$$

- **This tensor f_β has the following properties:**
  - f_β is a symplectic cocycle
  - $\beta \in Ker\ f_\beta$
  - The following symmetric tensor g_β , defined on all values of $ad_\beta(.)$ is positive definite:

$$g_\beta([\beta,Z_1],[\beta,Z_2])= f_\beta(Z_1,[\beta,Z_2])$$

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Souriau Lie Group Thermodynamic

$$f_\beta(Z_1,Z_2)= f(Z_1,Z_2)+Q.ad_{Z_1}(Z_2) \quad \text{with} \quad ad_{Z_1}(Z_2)=[Z_1,Z_2]$$

$$\beta \in Ker\ f_\beta \qquad g_\beta([\beta,Z_1],[\beta,Z_2])= f_\beta(Z_1,[\beta,Z_2])$$

- **Souriau equations are universal, because they are not dependent of the symplectic manifold but only of:**
  - the dynamical group ***G***
  - its symplectic cocycle f
  - the temperature β
  - the heat Q
- **Souriau called this model "Lie Groups Thermodynamics":**
  - "Peut-être cette thermodynamique des groups de Lie a-t-elle un intérêt mathématique".
- **For dynamic Galileo group (rotation and translation) with only one axe of rotation:**
  - this thermodynamic theory is the theory of centrifuge where the temperature vector dimension is equal to 2 (sub-group of invariance of size 2)
  - these 2 dimensions for vector-valued temperature are "thermic conduction" and "viscosity", unifying "heat conduction" and "viscosity".

Dynamic Gallileo Group

- **The Galileo group of an observer is the group of affine maps**

$$\begin{cases} \vec{x}' = R.\vec{x} + \vec{u}.t + \vec{w} \\ t' = t + e \end{cases}$$

$$\vec{x}, \vec{u} \text{ and } \vec{w} \in R^3, e \in R^+$$

$$R \in SO(3)$$

- **Matrix Form of Gallileo Group**

$$\begin{bmatrix} \vec{x}' \\ t' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \vec{u} & \vec{w} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ t \\ 1 \end{bmatrix}$$

- **Symplectic cocycles of the Galilean group: V. Bargmann (Ann. Math. 59, 1954, pp 1–46) has proven that the symplectic cohomology space of the Galilean group is one-dimensional.**

- **Lie Algebra of Gallileo Group** $\begin{bmatrix} \vec{\omega} & \vec{\eta} & \vec{\gamma} \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{cases} \vec{\eta} \text{ and } \vec{\gamma} \in R^3, \varepsilon \in R^+ \\ \vec{\omega} \in so(3) : \vec{x} \mapsto \vec{\omega} \times \vec{x} \end{cases}$

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Fundamental Souriau Theorem

- Let Ω be the largest open proper subset of $\mathfrak{g}$, Lie algebra of G, such that $\int_M e^{-\beta.U(\xi)} d\omega$ and $\int_M \xi.e^{-\beta.U(\xi)} d\omega$ are convergent integrals
- this set Ω is convex and is invariant under every transformation $\overline{a}_{\mathfrak{g}}$, where $a \mapsto \overline{a}_{\mathfrak{g}}$ is the adjoint representation of G, with:
  - $\beta \rightarrow \overline{a}_{\mathfrak{g}}(\beta)$
  - $\Phi \rightarrow \Phi - \theta\left(a^{-1}\right)\beta = \Phi + \theta(a).\overline{a}_{\mathfrak{g}}(\beta)$
  - $s \rightarrow s$
  - $Q \rightarrow \overline{a}_{\mathfrak{g}^*}(Q) + \theta(a) = \overline{a}_{\mathfrak{g}^*_\theta}(Q)$
  - $\varsigma \rightarrow \overline{a}_M^+(\varsigma)$
- where θ is the cocycle associated with the group G and the moment, and $\overline{a}_M^+(\varsigma)$ is the image under $\overline{a}_M$ of the probability measure ς .
- Rmq: Φ is changed but with linear dependence to β , then Fisher metric is unchanged by dynamical group:

$$I\left(\overline{a}_{\mathfrak{g}}(\beta)\right) = -\frac{\partial^2\left[\Phi - \theta\left(a^{-1}\right)\beta\right]}{\partial\beta^2} = -\frac{\partial^2\Phi}{\partial\beta^2} = I(\beta)$$

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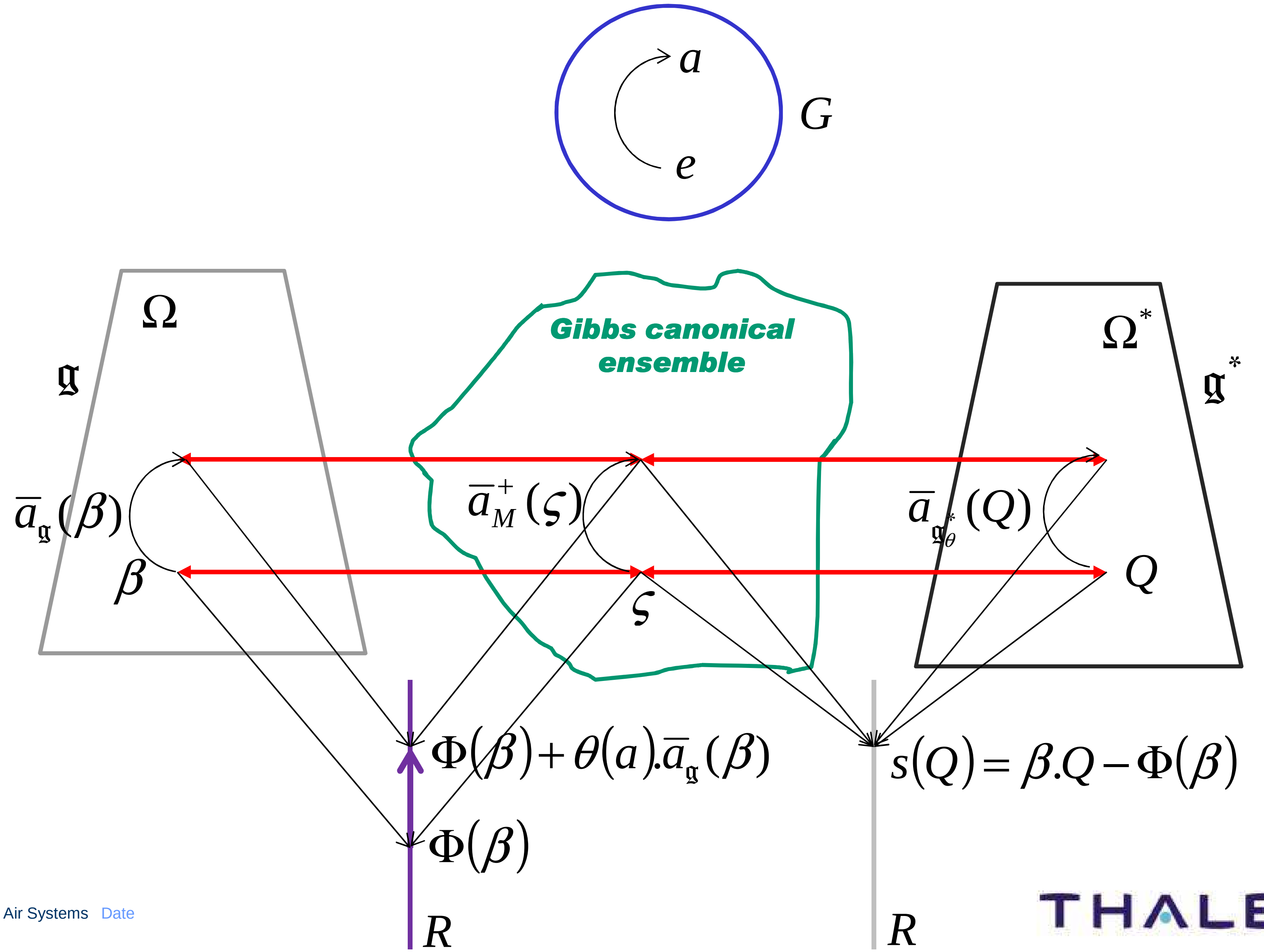
Fundamental Souriau Theorem



Souriau Lie Group definition of Fisher Metric

- Let f be the derivative of θ (symplectic cocycle of G) at the identity element and let us define:

$$\forall \beta \in \Omega, \quad f_\beta(Z_1, Z_2) = f(Z_1, Z_2) + Q.ad_{Z_1}(Z_2) \quad \text{with} \quad ad_{Z_1}(Z_2) = [Z_1, Z_2]$$

Then

- f_β is a symplectic cocycle of $\mathfrak{g}$, that is independent of the moment of G
- $f_\beta(\beta, \beta) = 0 \quad , \quad \forall \beta \in \Omega$
- There exists a symmetric tensor g_β defined on the image of $ad_\beta(.) = [., \beta]$ such that:

$$g_\beta([\beta, Z_1], Z_2) = f_\beta(Z_1, Z_2) \quad , \quad \forall Z_1 \in \mathfrak{g}, \ \forall Z_2 \in \mathrm{Im}(ad_\beta(.))$$

and

$$g_\beta(Z_1, Z_2) \geq 0 \quad , \quad \forall Z_1, Z_2 \in \mathrm{Im}(ad_\beta(.))$$

that gives the structure of a positive Euclidean space

Koszul Information Geometry, Souriau Lie Group Thermodynamics

	Koszul Information Geometry Model	***Souriau Lie Groups Thermodynamics Model***
Characteristic function	$\Phi(x) = -\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega$	$\Phi(\beta) = -\log \int_M e^{-\beta.U(\xi)} d\omega \quad \forall \beta \in \mathfrak{g}$
Entropy	$\Phi^*(x^*) = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi$	$s = -\int_M p(\xi) \log p(\xi) d\omega$
Legendre Transform	$\Phi^*(x^*) = \langle x, x^* \rangle - \Phi(x)$	$s(Q) = \beta.Q - \Phi(\beta)$
Density of probability	$p_x(\xi) = e^{-\langle x, \xi \rangle + \Phi(x)}$ $p_x(\xi) = \dfrac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}$	$p_\beta(\xi) = e^{-\beta.U(\xi) + \Phi(\beta)}$ $p_\beta(\xi) = \dfrac{e^{-\beta.U(\xi)}}{\int_M e^{-\beta.U(\xi)} d\omega}$
Dual Coordinate Systems	$x \in \Omega$ and $x^* \in \Omega^*$ $x^* = \int_{\Omega^*} \xi.p_x(\xi) d\xi = \dfrac{\int_{\Omega^*} \xi.e^{-\langle \xi, x \rangle} d\xi}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}$	$\beta \in \mathfrak{g}$ and $Q \in \mathfrak{g}^*$ $Q = \int_M U(\xi).p_\beta(\xi) d\omega = \dfrac{\int_M U(\xi) e^{-\beta.U(\xi)} d\omega}{\int_M e^{-\beta.U(\xi)} d\omega}$ β : Souriau Geometric Temperature U : Souriau Moment map Q : Mean of Souriau Moment Map or Geometric heat
Dual Coordinate Systems	$x^* = \dfrac{\partial \Phi(x)}{\partial x}$ and $x = \dfrac{\partial \Phi^*(x^*)}{\partial x^*}$	$Q = \dfrac{\partial \Phi}{\partial \beta}$ and $\beta = \dfrac{\partial s}{\partial Q}$
Hessian Metric	$ds^2 = -d^2 \Phi(x)$	$ds^2 = -d^2 \Phi(\beta)$
Fisher metric	$I(x) = -E_\xi \left[\dfrac{\partial^2 \log p_x(\xi)}{\partial x^2} \right]$ $I(x) = -\dfrac{\partial^2 \Phi(x)}{\partial x^2} = \dfrac{\partial^2 \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi}{\partial x^2}$	$I(\beta) = -E_\xi \left[\dfrac{\partial^2 \log p_\beta(\xi)}{\partial \beta^2} \right]$ $I(\beta) = -\dfrac{\partial^2 \Phi(\beta)}{\partial \beta^2} = \dfrac{\partial^2 \log \int_M e^{-\beta.U(\xi)} d\omega}{\partial \beta^2}$ $I(\beta) = -\dfrac{\partial^2 \Phi(\beta)}{\partial \beta^2} = -\dfrac{\partial Q}{\partial \beta}$ $K = -\dfrac{\partial Q}{\partial \beta}$: Souriau Geometric Capacity

Geometric heat Capacity / Specific heat

- We observe that the Information Geometry metric could be considered as a generalization of "Heat Capacity". Souriau called it the "Geometric Capacity". This geometric capacity is related to calorific capacity.

$$Q=\frac{\partial\Phi}{\partial\beta} \qquad I(\beta)=-\frac{\partial^2\Phi(\beta)}{\partial\beta^2}=-\frac{\partial Q}{\partial\beta}$$

$$\beta=\frac{1}{kT} \qquad K=-\frac{\partial Q}{\partial\beta}=-\frac{\partial Q}{\partial T}\left(\frac{\partial\frac{1}{kT}}{\partial T}\right)=\frac{1}{kT^2}\frac{\partial Q}{\partial T}$$

- Q is related to the mean, and K is related to the variance of U

$$Q=\frac{\partial\Phi}{\partial\beta}=\int_M U(\xi).p_\beta(\xi)d\omega=E_\xi[U]$$

$$I(\beta)=-\frac{\partial Q}{\partial\beta}=E_\xi\left[U^2\right]-E_\xi[U]^2=\int_M U(\xi)^2.p_\beta(\xi)d\omega-\left(\int_M U(\xi).p_\beta(\xi)d\omega\right)^2$$

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FOURIER HEAT EQUATION & HEAT CAPACITY

THÉORIE

ANALYTIQUE

DE LA CHALEUR,

PAR M. FOURIER.

A PARIS,

CHEZ FIRMIN DIDOT, PÈRE ET FILS,

LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE ET LA MARINE, RUE JACOB, N° 24.

1822.

Si l'on divise la quantité que l'on vient de trouver par celle qui est nécessaire pour élever la molécule de la température o à la température 1, on connaîtra l'accroissement de température qui s'opère pendant l'instant *dt*. Or, cette dernière quantité est C.D *dx dy dz*: car C désigne la capacité de chaleur de la substance; D sa densité, et *dx dy dz* le volume de la molécule. On a donc, pour exprimer le mouvement de la chaleur dans l'intérieur du solide, l'équation

$$\frac{dv}{dt} = \frac{K}{C.D}\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) \qquad (d)$$

D étant la densité du solide, ou le poids de l'unité de volume, et C la capacité spécifique, ou la quantité de chaleur qui élève l'unité de poids de la température o à la température 1; le produit C.D *dx dy dz* exprime combien il faut de chaleur pour élever de o à 1 la molécule dont le volume est *dx dy dz*. Donc en divisant par ce produit la nouvelle quantité de chaleur que la molécule vient d'acquérir, on aura son accroissement de température. On obtient ainsi

$$\frac{\partial Q}{\partial T} = C.D$$

$$I_{Fisher}(\beta) = -\frac{\partial Q}{\partial \beta} = \frac{1}{T^2}\frac{\partial Q}{\partial T}$$

$$\frac{\partial T}{\partial t} = \frac{\kappa}{C.D}\Delta T \Rightarrow \frac{\partial \beta^{-1}}{\partial t} = \kappa\left[I_{Fisher}(\beta)\right]^{-1}\beta^2\Delta\beta^{-1}$$

Θερμός : Souriau « Lie Group » Thermometer

Souriau has built a thermometer (θερμός) device principle that could measure the
Geometric Temperature
using "Relative Ideal Gas Thermometer" based on a theory of
Dynamical Group Thermometry
and has also recovered the
(Geometric) Laplace barometric law $p(\vec{r}) \propto e^{-m\beta\langle \vec{g},\vec{r} \rangle}$

$$s(Q) = \langle \beta, Q \rangle - \Phi(\beta) \qquad \beta = \frac{\partial s}{\partial Q} \in \mathfrak{g} \qquad Q = \frac{\partial \Phi}{\partial \beta} \in \mathfrak{g}^*$$

$$I_{Fisher}(\beta) = -\frac{\partial Q}{\partial \beta} = \mathrm{Var}[\mathrm{U}] \quad \text{with} \quad \begin{cases} Q : \text{Geometric Heat} \\ \beta : \text{Geometric (Planck) Temperature} \\ \mathrm{s} : \text{Geometric Entropy} \end{cases}$$

Koszul Information Geometry, Souriau Lie Group Thermodynamics

	Koszul Information Geometry Model	*Souriau Lie Groups Thermodynamics Model*
Convex Cone	$x \in \Omega$ Ω *convex cone*	$\beta \in \Omega$ Ω *convex cone: largest open subset of* $\mathfrak{g}$, *Lie algebra of G, such that* $\int_M e^{-\beta . U(\xi)} d\omega$ *and* $\int_M \xi . e^{-\beta . U(\xi)} d\omega$ *are convergent integrals*
Transformation	$x \rightarrow gx$ with $g \in Aut(\Omega)$	$\beta \rightarrow \bar{a}_{\mathfrak{g}}(\beta)$
Transformation of Potential (non invariant)	$\Phi_\Omega(x) \rightarrow \Phi_\Omega(gx) = \Phi_\Omega(x) + \log(\lvert\det g\rvert)$	$\Phi(\beta) \rightarrow \Phi(\bar{a}_{\mathfrak{g}}(\beta)) = \Phi(\beta) - \theta(a^{-1})\beta$
Transformation of Entropy (invariant)	$\Phi^*_{\Omega^*}(x^*) \rightarrow \Phi^*_{\Omega^*}\left(\frac{\partial \Phi_\Omega(gx)}{\partial x}\right) = \Phi^*_{\Omega^*}(x^*)$ with $x^* = \frac{\partial \Phi_\Omega(x)}{\partial x}$	$s(Q) \rightarrow s'(Q') = \beta'.Q' - \Phi' = \beta.Q - \Phi = s(Q)$ .with $\beta' = \bar{a}_{\mathfrak{g}}(\beta)$ $Q' = \frac{\partial \Phi'}{\partial \beta'} = \frac{\partial(\Phi + \theta(a)\bar{a}_{\mathfrak{g}}(\beta))}{\partial \bar{a}_{\mathfrak{g}}(\beta)} = \bar{a}_{\mathfrak{g}^*}(Q) + \theta(a)$ $\Phi' = \Phi(\beta') = \Phi(\bar{a}_{\mathfrak{g}}(\beta)) = \Phi(\beta) - \theta(a^{-1})\beta$
Information Geometry Metric (invariant)	$I(gx) = -\frac{\partial^2[\Phi_\Omega(x) + \log(\lvert\det g\rvert)]}{\partial x^2} = -\frac{\partial^2 \Phi_\Omega(x)}{\partial x^2} = I(x)$	$I(\bar{a}_{\mathfrak{g}}(\beta)) = -\frac{\partial^2[\Phi(\beta) - \theta(a^{-1})\beta]}{\partial \beta^2} = -\frac{\partial^2 \Phi(\beta)}{\partial \beta^2} = I(\beta)$

Invariance of Fisher Metric

- **In both Koszul and Souriau models, the Information Geometry Metric and the Entropy are invariant respectively to:**
  - the automosphisms g of the convex cone Ω
  - to $\overline{a}_{\mathfrak{g}}$ adjoint representation of Dynamical group G acting on Ω, the convex cone considered as largest open subset of $\mathfrak{g}$, Lie algebra of G, such that

$\int_M e^{-\beta.U(\xi)}d\omega$ and $\int_M \xi.e^{-\beta.U(\xi)}d\omega$ are convergent integrals.

$$x \rightarrow gx \quad \text{with} \quad g \in Aut(\Omega)$$

$$I(gx) = -\frac{\partial^2\left[\Phi_\Omega(x) + \log(\left|\det g\right|)\right]}{\partial x^2} = -\frac{\partial^2 \Phi_\Omega(x)}{\partial x^2} = I(x)$$

$$\beta \rightarrow \overline{a}_{\mathfrak{g}}(\beta)$$

$$I(\overline{a}_{\mathfrak{g}}(\beta)) = -\frac{\partial^2\left[\Phi(\beta) - \theta(a^{-1})\beta\right]}{\partial \beta^2} = -\frac{\partial^2 \Phi(\beta)}{\partial \beta^2} = I(\beta)$$

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Lie Group Action on Symplectic Manifold with Modern notation (cf. Charles-Michel Marle)

Lie Algebra of Lie Group

- **Let G a Lie Group and T_eG tangent space of G at its neutral element e**
  - Ad Adjoint representation of G

$$Ad : G \to GL(T_eG) \quad \text{with} \quad i_g : h \mapsto ghg^{-1}$$

$$g \in G \mapsto Ad_g = T_e i_g$$

  - ad Tangent application of Ad at neutral element e of G

$$ad = T_e Ad : T_eG \to End(T_eG) \qquad X, Y \in T_eG \mapsto ad_X(Y) = [X, Y]$$

- **For $G = GL_n(K)$ with $K = R$ or C**

$$T_eG = M_n(K) \qquad X \in M_n(K), g \in G \quad Ad_g(X) = gXg^{-1}$$

$$X, Y \in M_n(K) \quad ad_X(Y) = (T_e Ad)_X(Y) = XY - YX = [X, Y]$$

  - Curve from $e = I_d = c(0)$ tangent to $X = c(1)$: $c(t) = \exp(tX)$

    and transform by Ad : $\gamma(t) = Ad \exp(tX)$

$$ad_X(Y) = (T_e Ad)_X(Y) = \frac{d}{dt}\gamma(t)Y\Big|_{t=0} = \frac{d}{dt}\exp(tX)Y\exp(tX)^{-1}\Big|_{t=0} = XY - YX$$

Action of a Lie Group on a Symplectic Manifold

- **Let $\Phi : G \times M \to M$ be an action of Lie Group *G* on differentiable manifold *M*, the fundamental field associated to an element X of Lie algebra $\mathfrak{g}$ of group *G* is the vectors field X_M on *M*:**

$$X_M(x) = \frac{d}{dt}\Phi\left(\exp\left(-tX\right), x\right)\Big|_{t=0}$$

with $\Phi\left(g_1, \Phi(g_2, x)\right) = \Phi(g_1 g_2, x)$ **and** $\Phi(e, x) = x$

- **Φ is hamiltonian on a Symplectic Manifold M, if Φ is symplectic and if for all $X \in \mathfrak{g}$, the fundamental field X_M is globally hamiltonian**

- **There exist J_X linear application from $\mathfrak{g}$ to differential function on M**

$$\mathfrak{g} \to C^{\infty}(M, R)$$

$$X \to J_X$$

- **We can then associate a differentiable application J ,called moment of action Φ :**

$$J : M \to \mathfrak{g}^*$$

$$x \mapsto J(x) \text{ such that } J_X(x) = \left\langle J(x), X \right\rangle, X \in \mathfrak{g}$$

Action of a Lie Group on a Symplectic Manifold

- **We associate a bilinear and anti-symmetric form Θ , Symplectic Cocycle of Lie algebra $\mathfrak{g}$:**

  $\Theta(X,Y) = J_{[X,Y]} - \{J_X, J_Y\}$ **with** $\{.,.\}$: Poisson Bracket

  with $\Theta([X,Y],Z) + \Theta([Y,Z],X) + \Theta([Z,X],Y) = 0$

- **If $J' = J + \mu$ with constant $\mu \in \mathfrak{g}^*$, then:**

  $\Theta'(X,Y) = \Theta(X,Y) + \langle \mu, [X,Y] \rangle$

  With $\partial\mu(X,Y) = \langle \mu, [X,Y] \rangle$ **cobord of** $\mathfrak{g}$

Action of a Lie Group on a Symplectic Manifold

Equivariance of moment

- **There exist a unique affine action a such that linear part is coadjoint representation**

$$a : G \times \mathfrak{g}^* \to \mathfrak{g}^* \qquad \text{with } \left\langle Ad_g^* \xi, X \right\rangle = \left\langle \xi, Ad_{g^{-1}} X \right\rangle$$

$$a(g,\xi) = Ad_g^* \xi + \theta(g)$$

and that induce equivariance of moment J

$$J\left(\Phi(g,x)\right) = a(g, J(x)) = Ad_g^*\left(J(x)\right) + \theta(g)$$

$\theta : G \to \mathfrak{g}^*$ **is called Cocycle associated to J**

- **The differential $T_e\theta$ of 1-cocyle θ associated to J at neutral element e :**

$$\left\langle T_e\theta(X), Y \right\rangle = \Theta(X,Y) = J_{[X,Y]} - \left\{ J_X, J_Y \right\}$$

- **If $J' = J + \mu$ then :**

$$\Theta'(X,Y) = \Theta(X,Y) + \left\langle \mu, [X,Y] \right\rangle$$

$$\theta'(g) = \theta(g) + \mu - Ad_g^* \mu$$

Where $\mu - Ad_g^* \mu$ is cobord of G

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From Euler-Poincaré(-Tchesayev) Equation of Geometric Mechanics to Poincaré-Marle-Souriau Equation of Geometric Thermodynamics

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Seminal Paper of Poincaré 1901, revisited by Marle

[1] Henri Poincaré, Sur une forme nouvelle des équations de la Mécanique, C. R. Acad. Sci. Paris, T. CXXXII, n. 7, p. 369–371., 1901

- **Heni Poincaré proved that when a Lie algebra acts locally transitively on the configuration space of a Lagrangian mechanical system, the Euler-Lagrange equations are equivalent to a new system of differential equations defined on the product of the configuration space with the Lie algebra**

[2] C.-M. Marle, On Henri Poincaré's note on "Sur une forme nouvelle des équations de la Mécanique ", Journal of Geometry and Symmetries in Physics, JGSP 29, pp.1-38, 2013

- **Marle has written the Euler-Poincaré equations, under an intrinsic form, without any reference to a particular system of local coordinates**
- **Marle has proven that they can be conveniently expressed in terms of the Legendre and momentum maps of the lift to the cotangent bundle of the Lie algebra action on the configuration space**

Seminal Paper of Poincaré

« ***Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître*** »
Henri Poincaré, CRAS, 18 Février 1901

SÉANCE DU LUNDI 18 FÉVRIER 1901,

PRÉSIDENCE DE M. FOUQUÉ.

MEMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADÉMIE.

MÉCANIQUE RATIONNELLE. — *Sur une forme nouvelle des équations de la Mécanique*. Note de M. **H. Poincaré**.

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la Mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître.

$$\frac{d}{dt}\frac{dT}{d\eta_s} = \sum c_{ski}\frac{dT}{d\eta_i}\eta_k + \Omega_s.$$

« Elles sont surtout intéressantes dans le cas où U étant nul, T ne dépend que des η »
Henri Poincaré

THALES

EULER-POINCARE EQUATION by Lift of admissible Curve

- **Lagrangian Mechanical system with *M* is smooth configuration space**

$$I(\gamma)=\int_{t_0}^{t_1} L\left(\frac{d\gamma(t)}{dt}\right)dt \quad \text{with} \quad \gamma:[t_0,t_1]\rightarrow M \quad \text{and} \quad L:TM\rightarrow R$$

- **Poincaré assumes that a finite dimensional Lie algebra $\mathfrak{g}$ acts on the configuration manifold *M***

$$\varphi : M\times\mathfrak{g}\rightarrow TM \qquad \varphi^t : T^*M\rightarrow M\times\mathfrak{g}^*$$

$$(x,X)\mapsto\varphi(x,X)=X_M(x) \quad \text{(associated to Souriau moment map)}$$

- **New expression of the functional**

$$\bar{I}(\bar{\gamma})=\int_{t_0}^{t_1}\bar{L}\circ\bar{\gamma}(t)dt \quad \text{with} \quad \left|\begin{array}{l}\bar{\gamma}:[t_0,t_1]\rightarrow M\times\mathfrak{g}\\ t\mapsto\bar{\gamma}=(\gamma,V)\end{array}\right. \quad \text{and} \quad \left|\begin{array}{l}\bar{L}=L\circ\varphi:M\times\mathfrak{g}\rightarrow R\\ (x,X)\mapsto L(X_M(x))\end{array}\right.$$

$$\frac{d\gamma}{dt}=V(t),\ p_M\bar{\gamma}=\gamma \text{ with } p_M:M\times\mathfrak{g}\rightarrow M \text{ and } p_{\mathfrak{g}}\bar{\gamma}=V \text{ with } p_{\mathfrak{g}}:M\times\mathfrak{g}\rightarrow\mathfrak{g}$$

- **If γ is a parametrized continuous, piecewise differentiable curve in *M*, and $\bar{\gamma}$ any lift of γ *to M* × *g*, we have:**

$$\bar{I}(\bar{\gamma})=I(\gamma)$$

THALES

MARLE Intrinsic Expression of EULER-POINCARE EQUATION

- **J.M. Marle intrinsic expression (independant of any choice of local coordinates) of the Euler-Poincaré equation:**

$$\left(\frac{d}{dt}-ad^*_{V(t)}\right)\left(d_2\overline{L}\left(\gamma(t),V(t)\right)\right)=\Omega\left(\gamma(t),V(t)\right)$$

$$\Omega = p_{\mathfrak{g}^*}\circ\varphi^t\circ d_1\overline{L}:M\times\mathfrak{g}\to\mathfrak{g}^*$$

$$\begin{cases} d_1\overline{L}:M\times\mathfrak{g}\to T^*M \\ d_2\overline{L}:M\times\mathfrak{g}\to\mathfrak{g}^* \end{cases}$$ be the partial differentials of the function

$\overline{L}:M\times\mathfrak{g}\to R$ with respect to its 1st and its 2nd variable

$ad_V X=[V,X]=-[X,V]$ and $ad^*{}_V:\mathfrak{g}^*\to\mathfrak{g}^*$ such that

$\left\langle \xi, ad_V X\right\rangle=-\left\langle ad^*_V\xi, X\right\rangle$, $\xi\in\mathfrak{g}^*$, V and $X\in\mathfrak{g}$

$p_{\mathfrak{g}^*}:M\times\mathfrak{g}^*\to\mathfrak{g}^*$

- **Poincaré “This equation is useful mainly when $\overline{L}:M\times\mathfrak{g}\to R$ only depends on its second variable $X\in\mathfrak{g}$ (Lagrangian reduction)**

EULER-POINCARE-MARLE EQUATION

J.M. Marle has given the Euler-Poincaré Equation in terms of the Legendre and the Momentum Maps

- **Souriau Moment Map:** $J : T^*M \to \mathfrak{g}^*$

$$\langle J(\xi), X \rangle = \langle \xi, X_M \circ \pi_M(\xi) \rangle \ , \ \xi \in T^*M \ , \ X \in \mathfrak{g}$$

$$X_M \circ \pi_M(\xi) = \varphi(\pi_M(\xi), X)$$

$$\varphi^t : T^*M \to M \times \mathfrak{g}^*$$

$$\xi \mapsto \varphi^t(\xi) = (\pi_M(\xi), J(\xi))$$

$$J = p_{\mathfrak{g}^*} \circ \varphi^t$$

$$\varphi^t = (\pi_M, J)$$

- **The Legendre Map** $\mathbb{L}$

$$\mathbb{L} : TM \to T^*M \ (\text{vertical differential } d_{vert}L)$$

$$d_2\overline{L} = p_{\mathfrak{g}^*} \circ \varphi^t \circ \mathbb{L} \circ \varphi \quad \text{with} \quad J = p_{\mathfrak{g}^*} \circ \varphi^t$$

$$\Rightarrow d_2\overline{L} = J \circ \mathbb{L} \circ \varphi$$

- **Euler-Poincaré Equation with Legendre and Moment Maps:**

$$\left(\frac{d}{dt}-ad^*_{V(t)}\right)\left(J\circ\mathbb{L}\circ\varphi\left(\gamma(t),V(t)\right)\right)=J\circ d_1\bar{L}\left(\gamma(t),V(t)\right)$$

$$\frac{d\gamma(t)}{dt}=\varphi\left(\gamma(t),V(t)\right)$$

- **The Euler-Poincaré Equation and Reduction: Following the remark made by Poincaré at the end of his note, let us now assume that the map** $\bar{L}:M\times\mathfrak{g}\to R$ **only depends on its 2nd variable** $X\in\mathfrak{g}$

$$\left(\frac{d}{dt}-ad^*_{V(t)}\right)\left(d\bar{L}\left(V(t)\right)\right)=0$$

- **The Euler-Poincaré Equation in Hamiltonian Formalism**

$$H(\xi)=\left\langle \xi,\mathbb{L}^{-1}(\xi)\right\rangle-L\left(\mathbb{L}^{-1}(\xi)\right)\quad,\quad \xi\in T^*M$$

$$\mathbb{L}:TM\to T^*M$$

$$H:T^*M\to R$$

THALES

POINCARE-MARLE-SOURIAU EQUATION ON DUAL LIE ALGEBRA

- **Euler-Poincaré Equation in the Framework of Souriau Lie Group Thermodynamics**

$$\left(\frac{d}{dt}-ad^*_{V(t)}\right)\left(d\bar{L}\left(V(t)\right)\right)=0$$

- **Souriau Lie Group Thermodynamics:**

$$s(Q)=\langle\beta,Q\rangle-\Phi(\beta)=\langle\Theta^{-1}(Q),Q\rangle-\Phi\left(\Theta^{-1}(Q)\right)$$

$$\begin{cases} Q=\Theta(\beta)=\dfrac{\partial\Phi(\beta)}{\partial\beta}\in\mathfrak{g}^* \\ \beta=\Theta^{-1}(Q) \end{cases} \qquad \beta=\frac{\partial s(Q)}{\partial Q}\in\mathfrak{g}$$

- **Poincaré-Marle-Souriau Equation:**

$$\left(\frac{d}{dt}-ad^*_{\beta}\right)\left(\frac{\partial\Phi}{\partial\beta}\right)=0 \quad\Longrightarrow\quad \frac{dQ}{dt}=ad^*_{\beta}Q$$

THALES

POINCARE-MARLE-SOURIAU EQUATION ON DUAL LIE ALGEBRA

- **Recall that $\mathfrak{g}^*$ has a natural Poisson structure called Kirillov-Kostant-Souriau structure, which allow to associate to any smooth function $h:\mathfrak{g}^* \to R$ its Hamiltonian vector field :**

$$\chi_h(\xi) = -ad^*_{dh(\xi)}\xi \ , \ \xi \in \mathfrak{g}^*$$

- **If there exist a smooth function $h:\mathfrak{g}^* \to R$ such that $H = h \circ J$, the parametrized curve $\xi = J \circ \varsigma : [t_0, t_1] \to \mathfrak{g}^*$ satisfies the Hamilton differential equation on $\mathfrak{g}^*$:**

$$\frac{d\xi(t)}{dt} = -ad^*_{dh(\xi)}(\xi(t))$$

- **In Souriau Lie Group Thermodynamics:**

$$s(Q) = \langle \beta, Q \rangle - \Phi(\beta) \qquad ds = \beta dQ$$

$$\frac{dQ(t)}{dt} = -ad^*_{ds(Q(t))}(Q(t))$$

INFORMATION GEOMETRY BASED ON GEOMETRIC MECHANICS

GEOMETRIC MECHANICS
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-
SYMPLECTIC MECHANICS
J.M. SOURIAU

LIE GROUP THERMODYNAMICS
J.M. SOURIAU
-
HESSIAN GEOMETRY OF INFORMATION
J.L. KOSZUL

Élie CARTAN est le fils de Joseph CARTAN maréchal-ferrant.
Elie CARTAN le FORGERON
(du Latin FABER) l'Homme qui bat la matière sur l'ENCLUME, pour lui imprimer la COURBURE pour la mettre en FORME)

Texte de Bergson - Homo faber

"***En ce qui concerne l'intelligence humaine, on n'a pas assez remarqué que l'invention mécanique a d'abord été sa démarche essentielle****… . Si nous pouvions nous dépouiller de tout orgueil, si, pour définir notre espèce, nous nous en tenions strictement à ce que l'histoire et la préhistoire nous présentent comme la caractéristique constante de l'homme et de l'intelligence, nous ne dirions peut-être pas Homo sapiens,* ***mais Homo faber. En définitive, l'intelligence, envisagée dans ce qui en paraît être la démarche originelle, est la faculté de fabriquer des objets artificiels, en particulier des outils à faire des outils et d'en varier indéfiniment la fabrication****.*"

Henri Bergson, L' Évolution créatrice (1907), Éd. PUF, coll. "Quadrige", 1996, chap. II, pp.138-140

NEW FOUNDATION OF INFORMATION THEORY (Sapiens) by GEOMETRIC MECHANICS (Faber)

THALES

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Legendre Transform & Minimal Surface: Seminal Paper of Legendre

THALES

Legendre Transform and Minimal Surface

- **Legendre has introduced « Legendre Transform » to solve Minimal Surface Problem**

$$z(x,y) = p.x + q.y - \omega(p,q)$$

$$\text{with} \quad x = \frac{d\omega}{dp} \quad \text{and} \quad y = \frac{d\omega}{dq}$$

- **Classical “Legendre transform” with our previous notations:**

$$s(Q) = \beta.Q - \Phi(\beta) = \langle \beta, Q \rangle - \Phi(\beta)$$

$$\text{with} \quad \begin{cases} \Phi(\beta) = z(x,y) \\ s(Q) = \omega(p,q) \end{cases}, \quad \begin{cases} Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \\ \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \end{cases} \quad \text{and} \quad \begin{cases} Q = \begin{bmatrix} \dfrac{d\Phi}{d\beta_1} \\ \dfrac{d\Phi}{d\beta_2} \end{bmatrix} = \dfrac{d\Phi}{d\beta} \\ \beta = \begin{bmatrix} \dfrac{ds}{dQ_1} \\ \dfrac{ds}{dQ_2} \end{bmatrix} = \dfrac{ds}{dQ} \end{cases}$$

Legendre Transform and Minimal Surface

- In the following relation, we recover the definition of Entropy :

$$\begin{cases} x.dp + y.dq = d\omega \\ x = \dfrac{d\omega}{dp} \quad \text{and} \quad y = \dfrac{d\omega}{dq} \end{cases} \Rightarrow \begin{cases} \beta.dQ = ds \\ \beta = \dfrac{ds}{dQ} \end{cases}$$

2nd Principle of Thermodynamics

- But also relation with Mean Curvature

$$\frac{d}{d\beta}\left(\frac{Q}{\sqrt{1+\|Q\|^2}}\right) = \frac{d}{d\beta}\left(\frac{\dfrac{d\Phi}{d\beta}}{\sqrt{1+\left\|\dfrac{d\Phi}{d\beta}\right\|^2}}\right) = 2.H_\Phi$$

$$\|Q\|^2 << 1 \Rightarrow I(\beta) = -2H_\Phi$$

$$\|Q\|^2 >> 1 \Rightarrow \frac{d}{d\beta}\left(\frac{Q}{\|Q\|}\right) = \frac{\dfrac{dQ}{d\beta}.\|Q\| - Q.\dfrac{Q}{\|Q\|}}{\|Q\|^2} \Rightarrow \begin{cases} I(\beta) = -2.H_\Phi - \dfrac{1}{\|Q\|}\left(\dfrac{Q}{\|Q\|}\right)^2 \\ \text{with } \ Q = \dfrac{d\Phi}{d\beta} \end{cases}$$

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Past Conferences (GSI'13, MaxEnt'14) and Announcement (GSI'15 & Brillouin Seminar)

THALES

Past Conferences

- French/Indian Workshop on « **Matrix Information Geometry** », Ecole Polytechnique & Thales Research & Technology, 23-25th February 2011 (with Prof. Rajendra Bhatia)
  - http://www.lix.polytechnique.fr/~schwander/resources/mig/slides/
- SMAI'11 Congress, Mini-Symposium on « **Information Geometry** », 23-27th Mai 2011
  - http://smai.emath.fr/smai2011/programme_detaille.php
- Symposium on « **Information Geometry & Optimal Transport** », hosted at Institut Henri Poincaré, 12th February 2012 (with GDR CNRS MIA)
  - https://www.ceremade.dauphine.fr/~peyre/mspc/mspc-thales-12/
- SMF/SEE Conference on « **Geometric Science of Information** », Ecole des Mines de Paris, August 2013
  - http://www.see.asso.fr/gsi2013
- SMF/SEE Conference on MaxEnt with special Issue "Information, Entropy and their Geometric Structures", Clos Lucé in Amboise, sponsored by Jaynes Foundation, 21-26th Sept. 2014
  - https://www.see.asso.fr/node/10784
- Leon Brillouin Seminar on « **Geometric Science of Information** », Institut Henri Poincaré & IRCAM, launched by THALES since December 2009, with Ecole Polytechnque & INRIA/IRCAM
  - http://repmus.ircam.fr/brillouin/past-events

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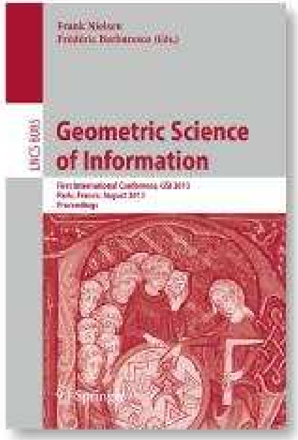
GSI'13 Proceedings



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F. Nielsen, F. Barbaresco (Eds.)

Geometric Science of Information

First International Conference, GSI 2013, Paris, France, August 28-30, 2013, Proceedings

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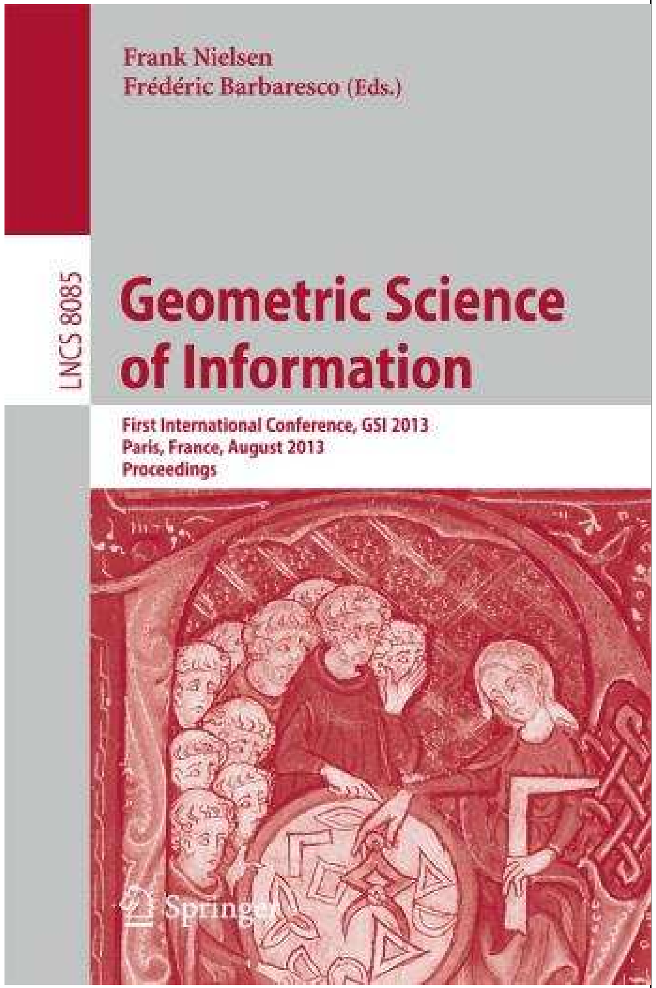

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THALES

MaxEnt'14, Amboise, Clos Lucé, September 2014

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Secretariat:
SEE MaxEnt 2014
SEE, France
Tel: (+33) 1 56 90 37 09
Fax: (+33) 1 56 90 37 19
Contact: http://www.see.asso.fr/node/9974

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First announcement and call for papers

The thirty-third **International Workshop on Bayesian Inference and Maximum Entropy Methods in Science and Engineering** will be held in France organized by SEE (http://www.see.asso.fr/en) under the auspices of « Centre national de la recherche scientifique (CNRS) », « Ecole supérieure d'électricité (SUPELEC) » and « Université Paris Sud (UPS), Orsay».

MaxEnt 2014 will take place in Château Le Clos Luce (http://www.vinci-closluce.com/en/) the last residence of Leonardo Da Vinci, in Amboise, France. Amboise is at the heart of Loire Valley Castles, labeled by UNESCO as World heritage. Le Clos Luce is at proximity of King Francis the First Amboise Castle, apex of the French Renaissance and of Vouvray wines vineyards and cellars.

MaxEnt 2014 strives to present Bayesian inference and Maximum Entropy methods in data analysis, information processing and inverse problems from a broad range of diverse disciplines: Astronomy and Astrophysics, Geophysics, Medical Imaging, Molecular Imaging and genomics, Non Destructive Evaluation, Particle and Quantum Physics, Physical and Chemical Measurement Techniques, Economics and Econometrics.

This year special interest will be Geometrical Sciences of Information / Information Geometry and their link with Entropy and their use with or without using Maximum Entropy and Bayesian inference. The focus will be more on using these concepts in generic Inverse problems, multidimensional and multi components Time Series Analysis and Spectral Estimation, Deconvolution and Source Separation, Segmentation, Classification and Pattern Recognition, X-ray, Diffractive, Diffusive and Quantum Tomographic Imaging.

List of provisional topics and organizers:

- History and axiomatic foundation of probability and Information theory
  (K.H. Knuth, A. Caticha, J. Skilling, ...)
- Information geometry and information theory
  (F. Barbaresco, F. Nielsen, H. Snoussi, C. Rodriguez, ...)
- Algorithms for Bayesian computation
  (A. Quinn, R. Fischer, ...)
- Bayesian Computed Tomography: medical imaging
  (Ch. Bouman, K. Sauer, E. Miller, J.M. Lina, ...)
- Non parametric Bayesian methods and experimental design
  (M. Jordan, ...)
- Bayesian classification, clustering, pattern recognition, image segmentation
  (Jun Zhang, ...)

The above list is not exclusive. It will be modified, completed and confirmed for the second call for papers.

The workshop includes a one day tutorial session, state of the art lectures, **invited papers, contributed papers, and poster presentations.**
Selected papers will be peer reviewed and published in American Institute of Physics (AIP) Proceedings series.
Contributed papers relating the above topics are being solicited.
Particularly encouraged are papers whose content is novel.

Abstracts (one page of about 400 words) of the proposed papers should be received by May 16, 2014

Tutorial day provisional topics and speakers:
Basics and history of Probability theory, Kevin H. Knuth
Basics and history of Information theory and Entropy, Ariel Caticha
Basics of Geometry and Manifolds, Frank Nielsen
Probability manifold and Geometry Carlos Rodriguez
Optimization and sampling, John Skilling
Bayesian and Information Geometry in signal processing Ali Mohammad-Djafari and F. Barbaresco

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http://djafari.free.fr/MaxEnt2014/Program_MaxEnt2014.html

Keynote speakers:
Misha Gromov (IHES, Abel Prize 2009)
Daniel Bennequin (Institut Mathématique de Jussieu)
Roger Balian (CEA, French Academy of science)
Stefano Bordoni (Bologna University)
https://www.see.asso.fr/node/10784

THALES

GSI’15 « Geometric Science of Information » 28th -30th October 2015 Ecole Polytechnique

GSI'15

GEOMETRIC SCIENCE OF INFORMATION

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SEE, France
Tel.: + 33 (0) 1 56 90 37 02
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First announcement and call for papers

As for GSI'13 (https://www.see.asso.fr/gsi2013), the objective of this SEE Conference GSI'15, hosted by Ecole Polytechnique, is to bring together pure/applied mathematicians and engineers, with common interest for Geometric tools and their applications for Information analysis.

It emphasizes an active participation of young researchers for deliberating emerging areas of collaborative research on "Information Geometry Manifolds and Their Advanced Applications".

Current and ongoing uses of Information Geometry Manifolds in applied mathematics are the following: Advanced Signal/Image/Video Processing, Complex Data Modeling and Analysis, Information Ranking and Retrieval, Coding, Cognitive Systems, Optimal Control, Statistics on Manifolds, Machine Learning, Speech/sound recognition, natural language treatment, etc., which are also substantially relevant for the industry.

The Conference will be therefore being held in areas of priority/focused themes and topics of mutual interest with a mandate to:

- Provide an overview on the most recent state-of-the-art
- Exchange of mathematical information/knowledge/expertise in the area
- Identification of research areas/applications for future collaboration
- Identification of academic & industry labs expertise for further collaboration

This conference will be an interdisciplinary event and will federate skills from Geometry, Probability and Information Theory. The conference proceedings are published in Springer's Lecture Note in Computer Science (LNCS) series.

Authors will be solicited to submit a paper in a special Issue "Geometric Science of Information" in Entropy Journal, an international and interdisciplinary open access journal of entropy and information studies published monthly online by MDPI.
http://www.mdpi.com/journal/entropy

Important Dates:

Main deadlines are the following:

- Deadline for 8 pages LNCS format: 1st of March 2015
- Notification of acceptance: 30th of April 2015
- Final paper submission: 30th of May 2015

Provisional topics of Special Sessions:

Provisional Topics of Special Sessions:

- Manifold/Topology Learning
- Riemannian Geometry in Manifold Learning
- Optimal Transport theory and applications in Imagery/Statistics
- Shape Space & Diffeomorphic mappings
- Geometry of distributed optimization
- Random Geometry/Homology
- Hessian Information Geometry
- Topology and Information
- Information Geometry Optimization
- Divergence Geometry
- Optimization on Manifold
- Lie Groups and Geometric Mechanics/Thermodynamics
- Quantum Information Geometry
- Infinite Dimensional Shape spaces
- Geometry on Graphs
- Bayesian and Information geometry for inverse problems
- Geometric structure of Audio Processing

Provisional program of Keynote Speakers:

Stochastic Flow on Compact and Non-Compact Manifolds: Marc Arnaudon
Geometric Mechanic and Shape, Tudor Ratiu *(To be Confirmed)*
To be completed

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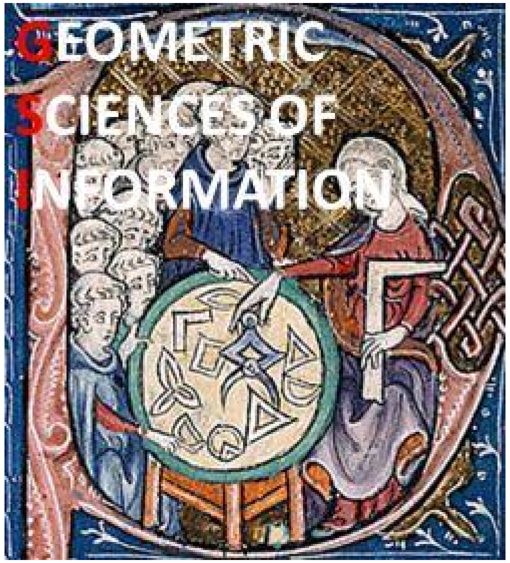

Authors will be solicited to submit a paper in a special Issue "Differential Geometrical Theory of Statistics” in ENTROPY Journal, an international and interdisciplinary open access journal of entropy and information studies published monthly online by MDPI.
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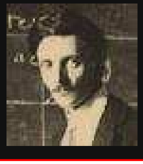


Séminaire Léon Brillouin

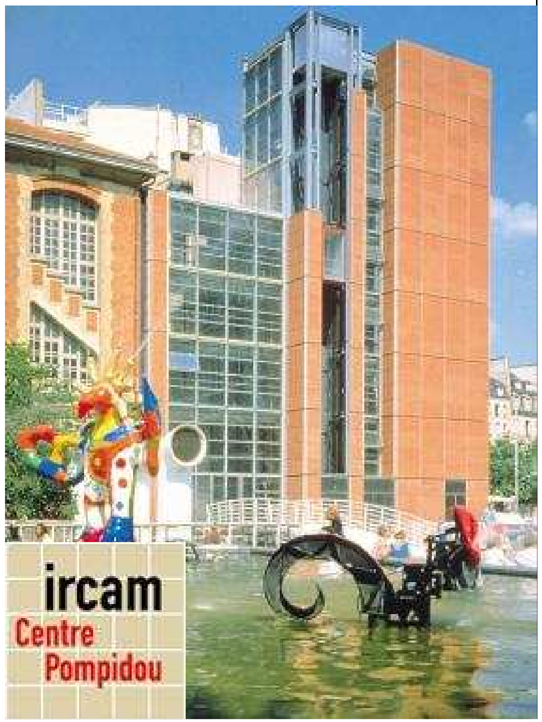
Sciences géométriques de l'information

http://repmus.ircam.fr/brillouin/home

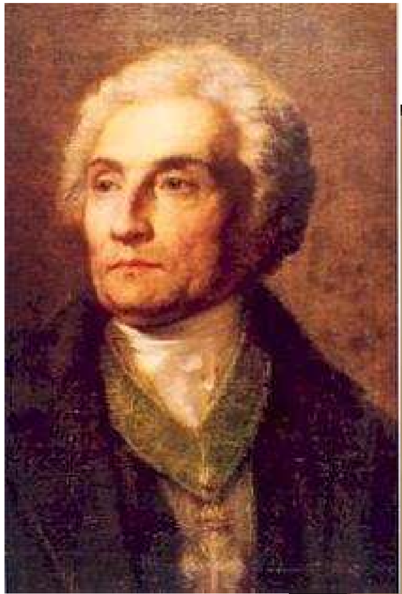
Léon Nicolas Brillouin Seminar

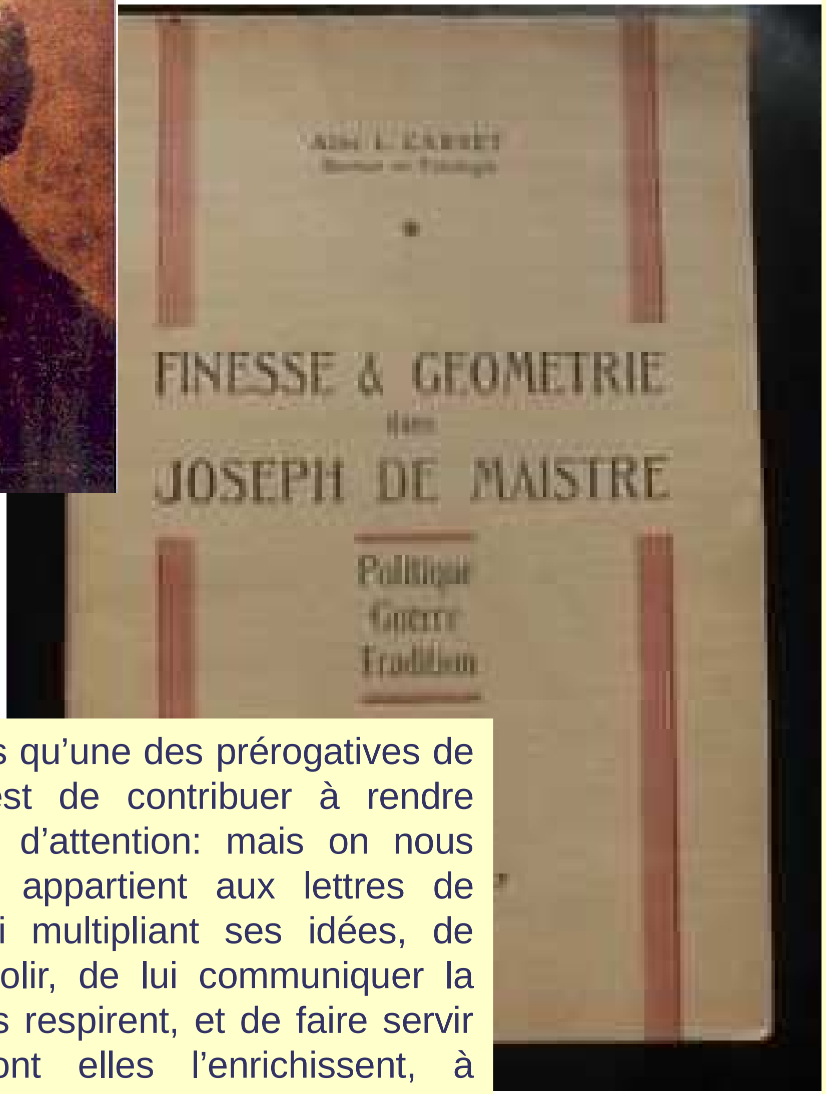
Corpus : « Geometric Science of Information »

- **Hosting lab:**
  - IRCAM, Salle Igor Stravinsky
  - Projet INRIA/CNRS/Ircam **MuSync** (Arshia Cont)
- **Animation : Arshia Cont (IRCAM & INRIA), Frank Nielsen (Ecole Polytechnique), F. Barbaresco (Thales)**
- **Web Site:** http://repmus.ircam.fr/brillouin/home
- **Abstracts, Videos & Slides:** http://repmus.ircam.fr/brillouin/past-events



Thank you for your attention





Nous avouerons qu'une des prérogatives de la géométrie est de contribuer à rendre l'esprit capable d'attention: mais on nous accordera qu'il appartient aux lettres de l'étendre en lui multipliant ses idées, de l'orner, de le polir, de lui communiquer la douceur qu'elles respirent, et de faire servir les trésors dont elles l'enrichissent, à l'agrément de la société.

Joseph de Maistre

Si on ajoute que la critique qui accoutume l'esprit, surtout en matière de faits, à recevoir de simples probabilités pour des preuves, est, par cet endroit, moins propre à le former, que ne le doit être la géométrie qui lui fait contracter l'habitude de n'acquiescer qu'à l'évidence; nous répliquerons qu'à la rigueur on pourrait conclure de cette différence même, que la critique donne, au contraire, plus d'exercice à l'esprit que la géométrie: parce que l'évidence, qui est une et absolue, le fixe au premier aspect sans lui laisser ni la liberté de douter, ni le mérite de choisir; au lieu que les probabilités étant susceptibles du plus et du moins, il faut, pour se mettre en état de prendre un parti, les comparer ensemble, les discuter et les peser. Un genre d'étude qui rompt, pour ainsi dire, l'esprit à cette opération, est certainement d'un usage plus étendu que celui où tout est soumis à l'évidence; parce que les occasions de se déterminer sur des vraisemblances ou probabilités, sont plus fréquentes que celles qui exigent qu'on procède par démonstrations: pourquoi ne dirions –nous pas que souvent elles tiennent aussi à des objets beaucoup plus importants ?

Joseph de Maistre