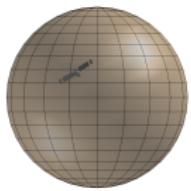
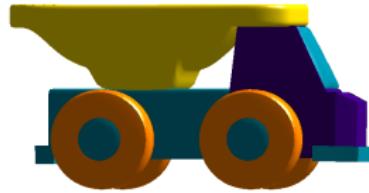
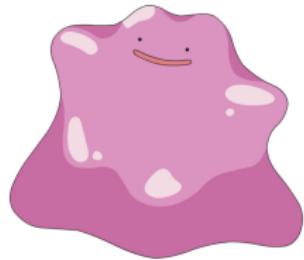


# About interpolation on manifolds...

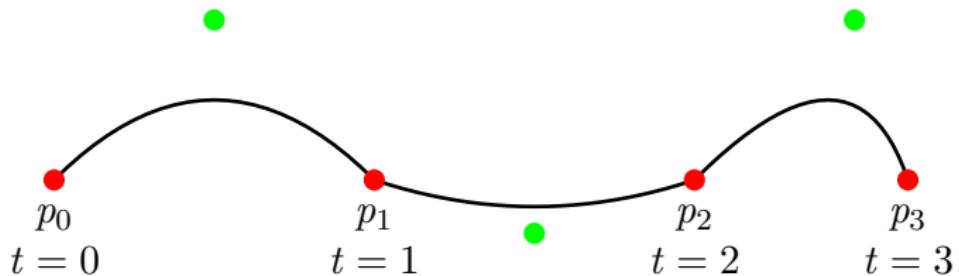


## How to interpolate points on curved spaces ?

Light      fast      general      good looking      interpolation

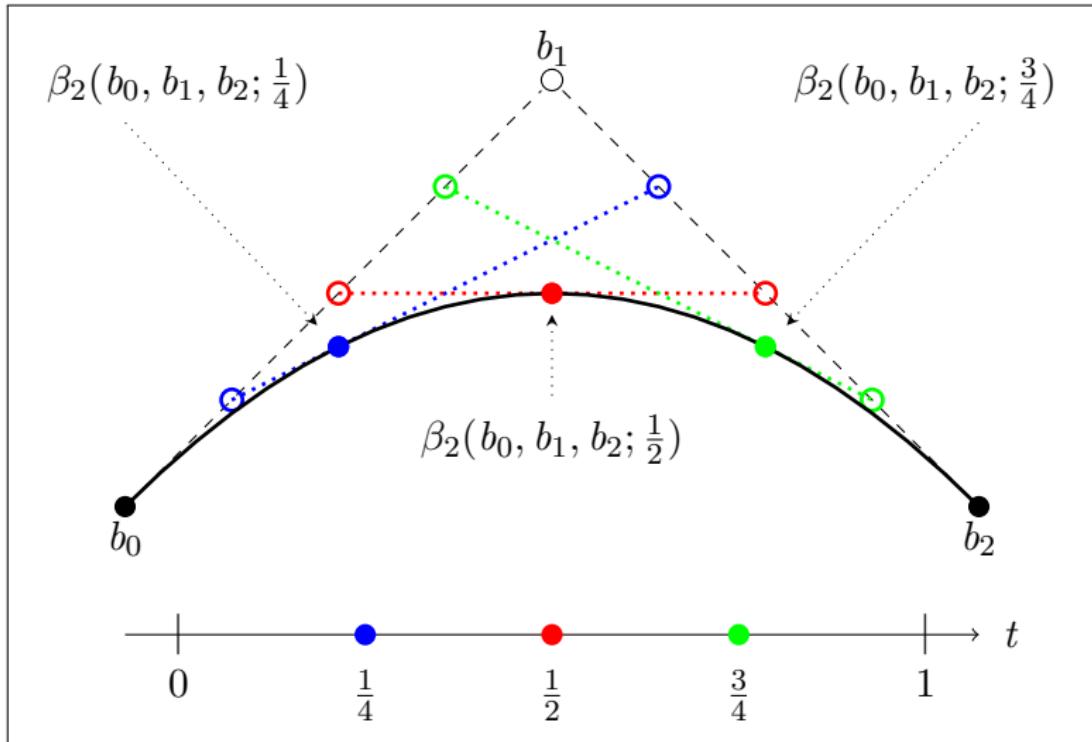
# How to interpolate?

Each segment between two consecutive points is a Bézier function.



**Light**      fast      general      good looking      **interpolation**

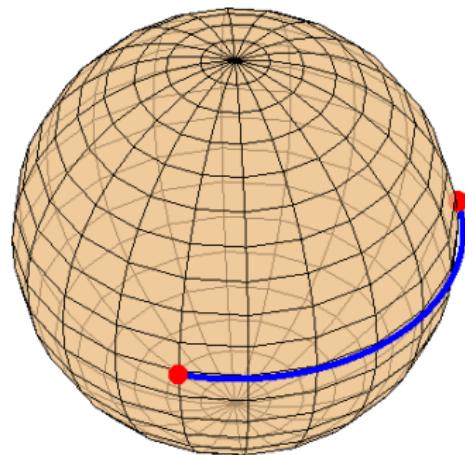
# Reconstruction : the De Casteljau algorithm



Light fast general good looking interpolation

# How to generalize Bézier curves to manifolds ?

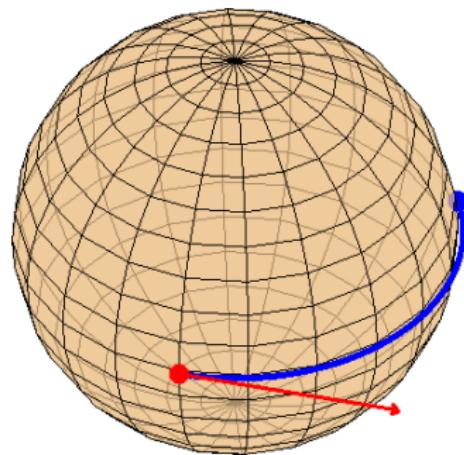
The straight line is a geodesic



# How to generalize Bézier curves to manifolds ?

The exponential map to construct the geodesic

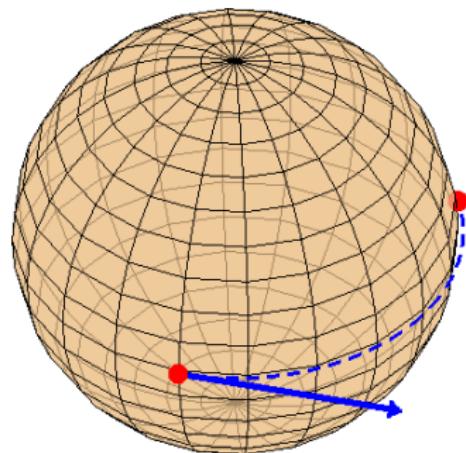
$$\gamma(t) = \text{Exp}_{\color{red}x}(t\xi_{\color{red}x})$$



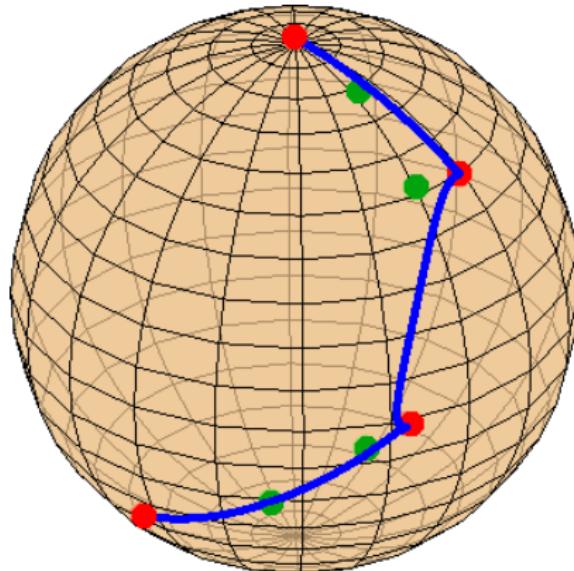
# How to generalize Bézier curves to manifolds ?

The logarithmic map to determine the starting velocity

$$\text{Log}_{\color{red}x}(y) = \xi_x$$



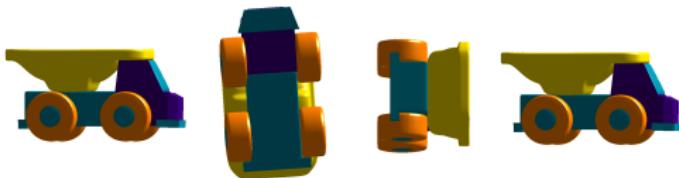
## Piecewise interpolation on the sphere



Light    fast    general    good looking    interpolation

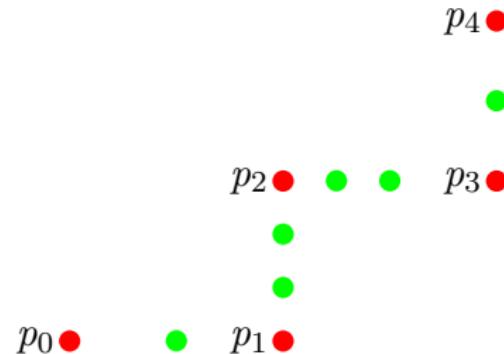
# Interpolation on Riemannian manifolds with a $\mathcal{C}^1$ piecewize-Bézier path

Pierre-Yves Gousenbourger



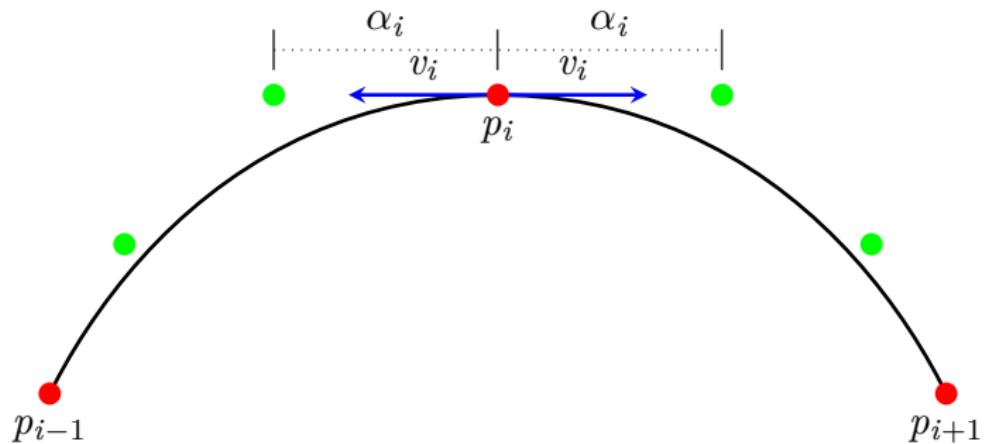
8 october 2014

## Good-looking curve on the Euclidean space



Find the optimal position of control points

## $\mathcal{C}^1$ -piecewise Bézier interpolation



$$b_i^L = \text{Exp}_{p_i}(-\alpha_i v_i)$$

$$b_i^R = \text{Exp}_{p_i}(-\alpha_i v_i)$$

# Optimal $\mathcal{C}^1$ -piecewise Bézier interpolation

Minimization of the mean square acceleration of the path

$$\underbrace{\min_{\alpha_i} \int_0^1 \|\ddot{\beta}_2^0(\alpha_i; t)\|^2 dt + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i(\alpha_i; t)\|^2 dt + \int_0^1 \|\ddot{\beta}_2^n(\alpha_i; t)\|^2 dt}_{\text{Second order polynomial } P(\alpha_i)}$$

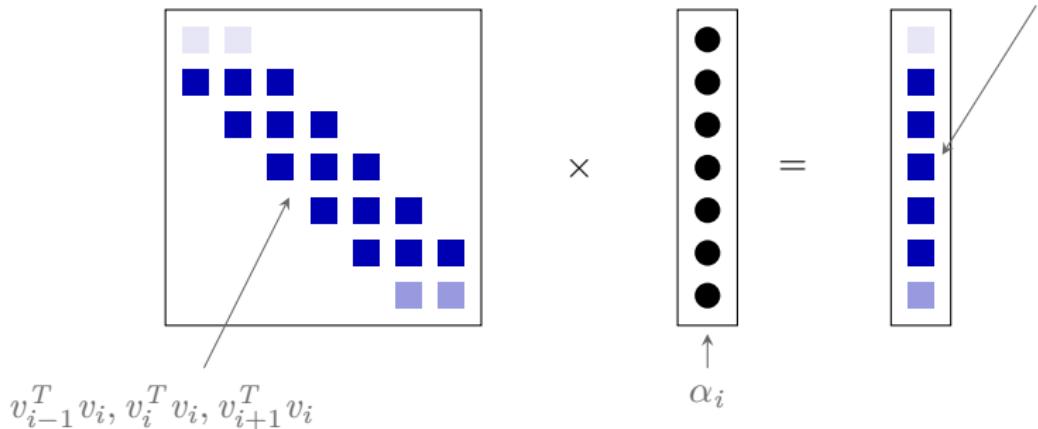
$$\nabla P(\alpha_i) !$$

# Optimal $\mathcal{C}^1$ -piecewise Bézier interpolation

Minimization of the mean square acceleration of the path

$$\underbrace{\min_{\alpha_i} \int_0^1 \|\ddot{\beta}_2^0(\alpha_i; t)\|^2 dt + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i(\alpha_i; t)\|^2 dt + \int_0^1 \|\ddot{\beta}_2^n(\alpha_i; t)\|^2 dt}_{\text{Second order polynomial } P(\alpha_i)}$$

$$\sim (p_{i-1} - p_i)^T v_i$$

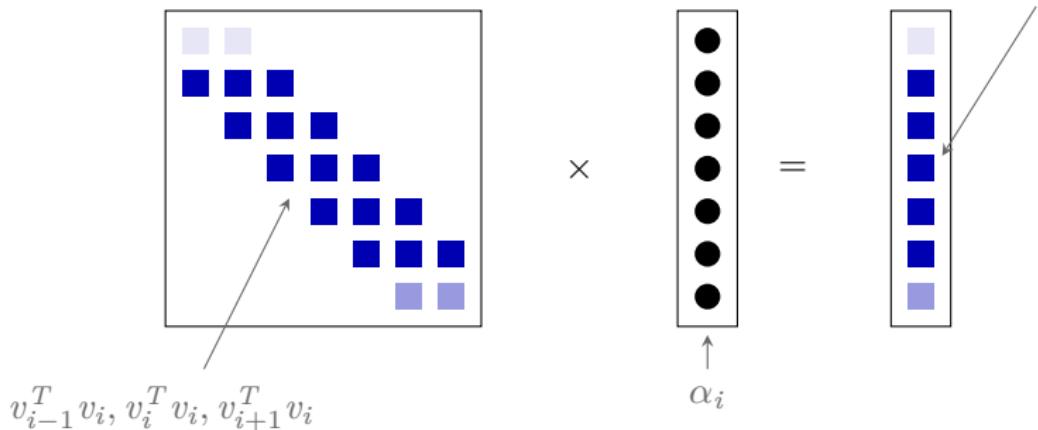


# Optimal $\mathcal{C}^1$ -piecewise Bézier interpolation

Minimization of the mean square acceleration of the path

$$\underbrace{\min_{\alpha_i} \int_0^1 \|\ddot{\beta}_2^0(\alpha_i; t)\|^2 dt + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i(\alpha_i; t)\|^2 dt + \int_0^1 \|\ddot{\beta}_2^n(\alpha_i; t)\|^2 dt}_{\text{Second order polynomial } P(\alpha_i)}$$

$$\sim (p_{i-1} - p_i)^T v_i$$



# Optimal $\mathcal{C}^1$ -piecewise Bézier interpolation

Minimization of the mean square acceleration of the path

$$\underbrace{\min_{\alpha_i} \int_0^1 \|\ddot{\beta}_2^0(\alpha_i; t)\|^2 dt + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i(\alpha_i; t)\|^2 dt + \int_0^1 \|\ddot{\beta}_2^n(\alpha_i; t)\|^2 dt}_{\text{Second order polynomial } P(\alpha_i)}$$

$$\sim (\text{Log}_{p_i}(p_{i-1}))^T v_i$$

$$\begin{array}{c} \text{Diagram showing matrix multiplication:} \\ \begin{array}{ccc} \begin{matrix} & \text{purple} & \text{purple} \\ \text{blue} & \text{blue} & \text{blue} \\ & \text{blue} & \text{blue} \end{matrix} & \times & \begin{matrix} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{matrix} \\ \text{Matrix} & & \text{Vector} \end{array} = \begin{matrix} & \text{purple} \\ \text{blue} & \text{blue} & \text{blue} \\ & \text{blue} & \text{blue} \end{matrix} \\ \text{Resulting Matrix} \end{array}$$

$v_{i-1}^T v_i, v_i^T v_i, v_{i+1}^T v_i$

$\alpha_i$

# Optimal $\mathcal{C}^1$ -piecewise Bézier interpolation

Minimization of the mean square acceleration of the path

$$\underbrace{\min_{\alpha_i} \int_0^1 \|\ddot{\beta}_2^0(\alpha_i; t)\|^2 dt + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i(\alpha_i; t)\|^2 dt + \int_0^1 \|\ddot{\beta}_2^n(\alpha_i; t)\|^2 dt}_{\text{Second order polynomial } P(\alpha_i)}$$

$$\sim (\text{Log}_{p_i}(p_{i-1}))^T v_i$$

The diagram shows a matrix multiplication operation:

$$\begin{matrix} & \times & = & \\ \begin{matrix} \text{Matrix } A \\ \text{Dimensions: } 6 \times 3 \end{matrix} & \times & \begin{matrix} \text{Vector } b \\ \text{Dimensions: } 3 \times 1 \end{matrix} & = & \begin{matrix} \text{Resultant Vector } c \\ \text{Dimensions: } 6 \times 1 \end{matrix} \end{matrix}$$

Matrix  $A$  is a 6x3 matrix represented by a grid of colored squares (blue, dark blue, light blue) with arrows indicating row and column indices:

Light Blue	Light Blue	
Dark Blue	Dark Blue	Dark Blue
Dark Blue	Dark Blue	Dark Blue
Dark Blue	Dark Blue	Dark Blue
Dark Blue	Dark Blue	Dark Blue
Light Blue	Light Blue	

Vector  $b$  is a 3x1 column vector represented by a vertical stack of black dots:

$$b = \begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{pmatrix}$$

Resultant Vector  $c$  is a 6x1 column vector represented by a vertical stack of colored squares (blue, dark blue, light blue) with an arrow pointing to index  $i$ :

$$c = \begin{pmatrix} \text{Light Blue} \\ \text{Dark Blue} \\ \text{Dark Blue} \\ \text{Dark Blue} \\ \text{Dark Blue} \\ \text{Light Blue} \end{pmatrix}$$

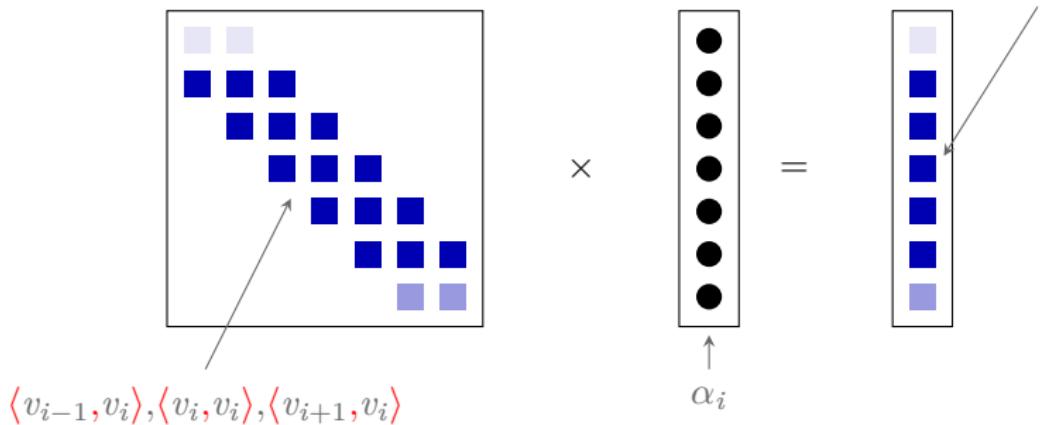
Below the matrix  $A$ , the expression  $v_{i-1}^T v_i, v_i^T v_i, v_{i+1}^T v_i$  is shown, where  $v_i$  is the  $i$ -th column of  $A$ .

# Optimal $\mathcal{C}^1$ -piecewise Bézier interpolation

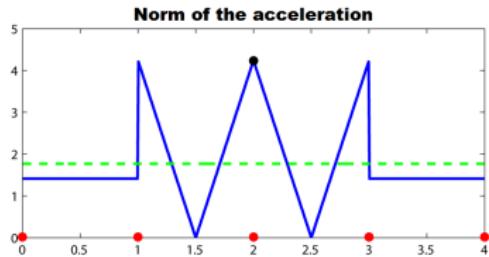
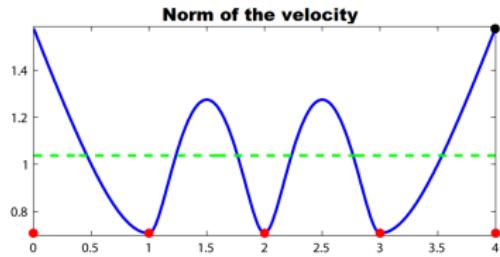
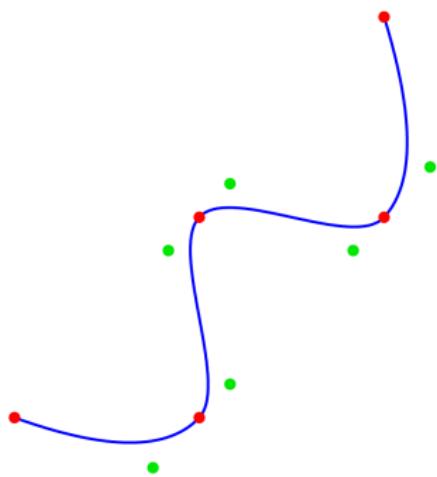
Minimization of the mean square acceleration of the path

$$\underbrace{\min_{\alpha_i} \int_0^1 \|\ddot{\beta}_2^0(\alpha_i; t)\|^2 dt + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i(\alpha_i; t)\|^2 dt + \int_0^1 \|\ddot{\beta}_2^n(\alpha_i; t)\|^2 dt}_{\text{Second order polynomial } P(\alpha_i)}$$

$$\sim \langle \text{Log}_{p_i}(p_{i-1}), v_i \rangle$$

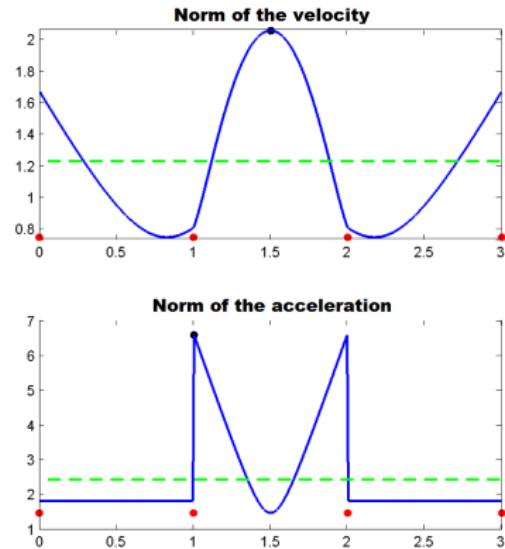
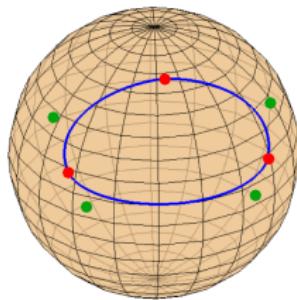


## A result on $\mathbb{R}^2$

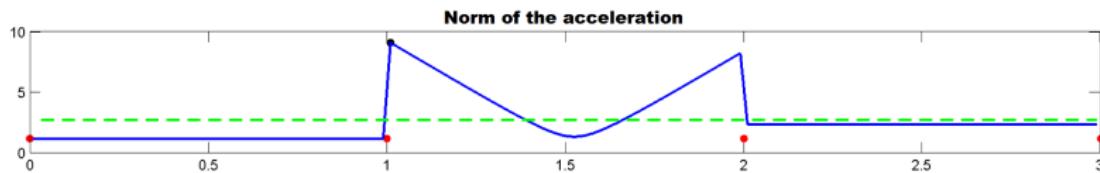
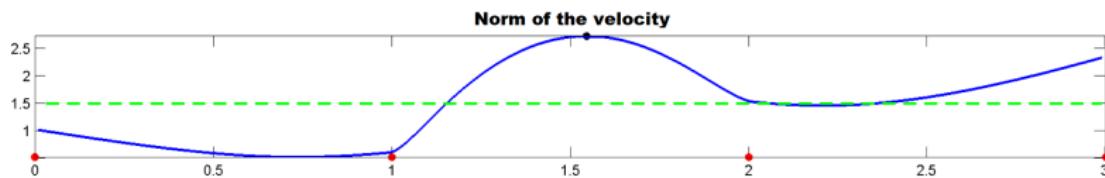


Light fast general good looking interpolation

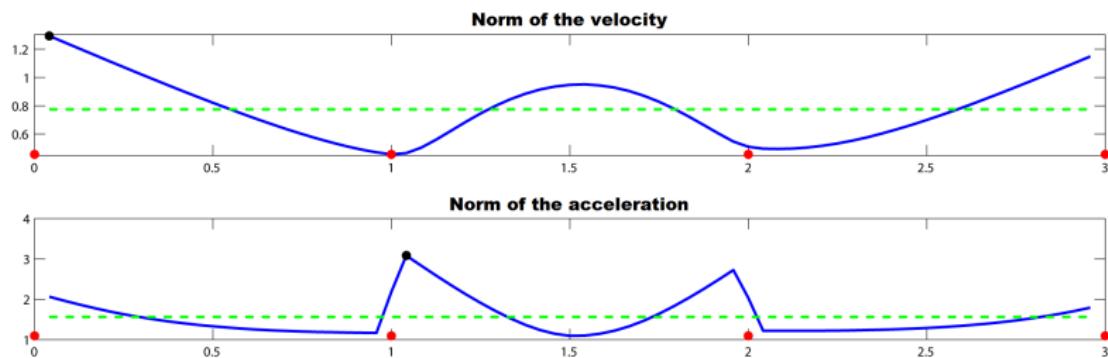
# Generalization to manifolds : the sphere $\mathbb{S}^2$



# Generalization to manifolds : the special orthogonal group $SO(3)$



# Generalization to manifolds : morphing of shapes



## Conclusions

Light fast general good looking interpolation

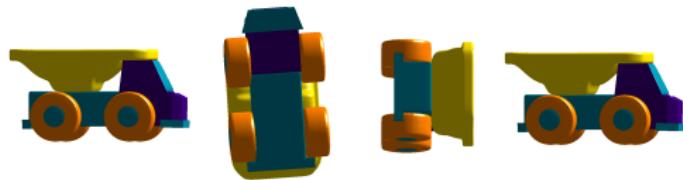
No choice of velocities  $v_i$ ? (Arnould, Samir, Absil)

Application to manifolds of high dimension?

Any questions ?

# Interpolation on Riemannian manifolds with a $\mathcal{C}^1$ piecewize-Bézier path

Pierre-Yves Gousenbourger



8 october 2014