Geodesic shooting on shape spaces

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Outline

Riemannian manifolds (finite dimensional)

Spaces spaces (intrinsic metrics)

Diffeomorphic transport and homogeneous shape spaces

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Riemannian geometry

The classical apparatus of (finite dimensional) *riemannian geometry* starts with the definition of a **metric** $\langle\ ,\ \rangle_m$ on the tangent bundle.

Geodesics and energy

Find the path $t \to \gamma(t)$ from m_0 to m_1 minimizing the energy

$$I(\gamma) \doteq \int_0^1 \langle \dot\gamma(t), \dot\gamma(t) \rangle_{\gamma(t)} dt$$

Critical paths from I are **geodesics**

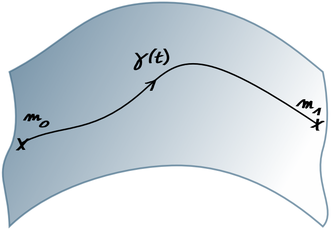

Figure: **Path** $\gamma(t)$

Geodesic equation

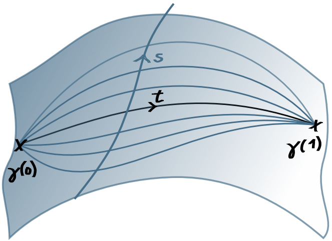

Figure: **Variations** around $\gamma(t)$

$$\frac{dl}{ds}(\gamma) = -\int_0^T \langle \frac{D}{\partial t}\dot{\gamma}, \frac{\partial}{\partial s}\gamma \rangle_{\gamma(t)} dt$$

$$\delta l \equiv 0 \text{ for } \frac{D}{dt}\dot{\gamma} \equiv 0$$

where $\frac{D}{dt} = \nabla_{\dot{\gamma}}$ is the **covariant derivative** along γ

Second order EDO given $\gamma(0), \dot{\gamma}(0)$.

Exponential Mapping and Geodesic Shooting

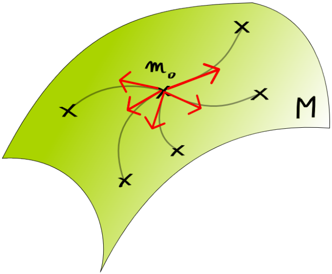

Figure: Exponential mapping and normal cordinates

This leads to the definition of the **exponential mapping**

$$Exp_{\gamma(0)} : T_{\gamma(0)}M \to M \, .$$

Starts at $m_0 = \gamma(0)$, chooses the direction $\gamma'(0) \in : T_{\gamma(0)}M$ and **shoots** along the geodesic to $m_1 = \gamma(1)$.

Key component of many interesting problems : Generative models, Karcher means, parallel transport via Jacobi fields, etc.

Lagrangian Point of View

(In local coordinates)

- **Constrained minimization problem**

$$\begin{cases} \int_0^1 L(q(t), \dot{q}(t))dt \\ \\ \text{with Lagrangian } L(q, \dot{q}) = \frac{1}{2}|\dot{q}|_q^2 = \frac{1}{2}(L_q\dot{q}|\dot{q}) \\ \text{ and } (q_0, q_1) \text{ fixed} \end{cases}$$

L_q codes the metric. L_q symmetric positive definite.

- **Euler-Lagrange equation**

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

From Lagrangian to Hamiltonian Variables

- Change $(q, \dot{q})$ (position, velocity) $\rightarrow (q, p)$ (position, momentum) with

$$p = \frac{\partial L}{\partial \dot{q}} = L_q \dot{q}$$

- Euler-Lagrange equation is equivalent to the **Hamiltonian equations** :

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p}(q, p) \\ \\ \dot{p} = -\frac{\partial H}{\partial q}(q, p) \end{cases}$$

where (Pontryagin Maximum Principle)

$$H(q, p) \doteq \max_u \, (p|u) - L(q, u) = \frac{1}{2}(K_q p | p)$$

$K_q = L_q^{-1}$ define the co-metric.

Note: $\partial_q H$ induces the derivative of K_q with respect to q.

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Parametrized shapes

The *ideal* mathematical setting: A smart space $\mathcal{Q}$ of smooth mappings from a smooth manifold S to $\mathbb{R}^d$.

Basic spaces are $\mathrm{Emb}(S, \mathbb{R}^d)$, $\mathrm{Imm}(S, \mathbb{R}^d)$ the space of smooth (say C^∞) embeddings or immersions from S to $\mathbb{R}^d$. May introduce a finite regularity $k \in \mathbb{N}^*$ and speak about $\mathrm{Emb}^k(S, \mathbb{R}^d)$ and $\mathrm{Imm}^k(S, \mathbb{R}^d)$.

$S = S^1$ for close curves, $S = S^2$ for close surfaces homeomorphic to the sphere

Nice since open subset of $C^\infty(S, \mathbb{R})$. For $k > 0$, open subset of a Banach space.

Metrics

Case of curves: S^1 is the unit circle.

- L^2 metric : $h, h' \in T_q C^\infty(S, \mathbb{R}^d)$

$$\langle h, h' \rangle_q = \int \langle h, h' \rangle |\partial_\theta q| d\theta = \int_{S^1} \langle h, h' \rangle ds\,.$$

  - Extensions in Michor and Mumford (06)

- $\dot{H}^1$ type metric :

$$\langle h, h' \rangle_q = \int_{S^1} \langle (D_s h)^\perp, (D_s h')^\perp \rangle + b^2 \langle (D_s h)^\top, (D_s h')^\top \rangle ds$$

  where $D_s = \partial_\theta / |\partial_\theta q|$

  - Younes's elastic metric (*Younes '98*, $b = 1$, $d = 2$), *Joshi Klassen Srivastava Jermyn '07* for $b = 1/2$ and $d \geq 2$ (SRVT trick).

Metrics (Cont'd)

Parametrization invariance: $\psi \in \mathrm{Diff}(S)$

$$\langle h \circ \psi, h' \circ \psi \rangle_{q \circ \psi} = \langle h, h' \rangle_q \,.$$

- Sobolev metrics (*Michor Mumford '07; Charpiat Keriven Faugeras '07; Sundaramoorthi Yezzi Mennuci '07*): $a_0 > 0$, $a_n > 0$

$$\langle h, h' \rangle_q = \int_{S^1} \sum_{i=0}^{n} a_i \langle D_s^i h, D_s h' \rangle \, ds \,.$$

  Again, paramerization invariant metric.
- Extension for surfaces ($\dim(S) \geq 2$) in *Bauer Harms Michor '11*.

Summary and questions

- Many possible metrics on the preshape spaces $\mathcal{Q}$ (how to choose)
- Ends up with a smooth parametrization invariant metric on a smooth preshape space $\mathcal{Q}$ and a riemmanian geodesic distance.

Questions: Minimal: Local existence of geodesic equations and smoothness for smooth data ? More

1. Existence of global solution (in time) of the geodesic equation (geodesically complete metric space) ?
2. Existence of a minimising geodesic between any two points (geodesic metric space) ?
3. Completeness of the space for the geodesic distance (complete metric space) ?

1-2-3 equivalents on finite dimensional riemannian manifold (Hopf-Rinow thm)

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Few answers

- (Local solution): Basically, for Sobolev norm of order n greater than 1, local existence of solutions of the geodesic equation if the intial data has enough regularity (*Bauer Harms Michor '11*):

$$k > \frac{\dim(S)}{2} + 2n + 1$$

- (Geodesic completeness): Global existence has been proved recently for $S = S^1$, $d = 2$ (planar shapes) and $n = 2$ (*Bruveris Michor Mumford '14*). Wrong for the order 1 Sobolev metric. Mostly unkown for the other cases.
- (Geodesic metric spaces): Widely open
- (Complete metric space): No for smooth mappings (weak metric). Seems to be open for $\mathrm{Imm}^k(S, \mathbb{R}^d)$ or $\mathrm{Emb}^k(S, \mathbb{R}^d)$ and order k Sobolev metric.

Why there is almost no free lunch

Back to the Hamiltonian point of view.
The metric can be written $(L_q h|h)$ with L_h an elliptic symmetric definite diferential operator.

$$H(q,p) = \frac{1}{2}(K_q p|q)$$

where $K_q = L_q^{-1}$ is a pseudo-differential operator with a really intricate dependency with the pre-shape q.

Towards shape shapes: removing parametrisation

Diff(S) as a nuisance parameter

- Diff(S): the diffeomorphism group on S (reparametrization).
- Canonical shape spaces : $\mathrm{Emb}(S, \mathbb{R}^d)/\mathrm{Diff}(S)$ or $\mathrm{Imm}(S, \mathbb{R}^d)/\mathrm{Diff}(S)$

$$[q] = \{q \circ \psi \mid \psi \in \mathrm{Diff}(S)\}$$

- Structure of manifold for $\mathrm{Emb}(S, \mathbb{R}^d)/\mathrm{Diff}(S)$ and $\mathrm{Imm}(S, \mathbb{R}^d)/\mathrm{Diff}(S)$ (orbifold)
- Induced geodesic distance

$$d_{\mathcal{Q}/\mathrm{Diff}(S)}([q_0], [q_1]) = \inf\{d_{\mathcal{Q}}(q_0, q_1 \circ \psi) \mid \psi \in \mathrm{Diff}(S)\ \}$$

Questions: Given to two curves q_0 and q_{targ} representing two shapes $[q_0]$ and $[q_{targ}]$

- Existence of an *horizontal* geodesic path $t \mapsto q_t \in \mathcal{Q}$ emanating from q_0 and of a reparametrisation path $t \mapsto \psi_t \in \text{Diff}(S)$ such that $q_{targ} = q_1 \circ \psi_1$?

No available shooting algorithms for parametrized curves or surfaces, only mainly path straightening algorithms or DP algorithms that alternate between q and ψ.

Usually, no guarantee of existence of an optimal diffeomorphic parametrisation ψ_1 (T. Younes '97).

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Shape spaces as homogeneous spaces

Idea #1:

D'Arcy Thomspon and Grenander. Put the emphasis on the **left** action of the group of diffeomorphisms on the embedding space $\mathbb{R}^d$ and consider homogeneous spaces $M = G.m_0$:

$$G \times M \to M$$

Diffeomorphisms can act on almost everything
(changes of coordinates)!

Idea #2:

Put the metric on the group G (right invariance). More simple. Just need to specify the metric at the identity.

Shape spaces as homogeneous spaces (Cont'd)

Idea #3:

Build the metric on M from the metric on G :

1. If G has a G (right)-equivariant metric :

   $$d_G(g_0 g, g_0 g') = d_G(g, g') \text{ for any } g_0 \in G$$

   then M inherits a quotient metric

   $$d_M(m_0, m_1) = \inf\{\, d_G(\mathrm{Id}, g) \mid g m_0 = m_1 \in G\}$$

2. The geodesic on Gm_0 can be lifted to a geodesic in G (horizontal lift).

Construction of right-invariant metrics

Start from a Hilbert space $V \hookrightarrow C_0^1(\mathbb{R}^d, \mathbb{R}^d)$.

1. Integrate time dependent vector fields $v(.) = (v(t))_{t\in[0,1]}$:

$$\dot{g} = v \circ g,\ g(0) = \mathrm{Id}\,.$$

2. Note $g^v(.)$ the solution and

$$G_V \doteq \{\ g^v(1) \mid \int_0^1 |v(t)|_V^2 dt < \infty\ \}\,.$$

$$d_{G_V}(g_0, g_1) \doteq \left(\inf\{\int_0^1 |v(t)|_V^2 dt < \infty \mid g_1 = g^v(1) \circ g_0\ \}\right)^{1/2}$$

Basic properties

Thm (T.)

If $V \hookrightarrow C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ then

1. *G_V is a **group** of C^1 diffeomorphisms on $\mathbb{R}^d$.*
2. *G_V is a **complete metric space** for d_G*
3. *we have existence of a minimizing geodesic between any two group elements g_0 and g_1 (**geodesic metric space**)*

Note: G_V is parametrized by V which is not a Lie algebra. Usualle G_V andd_G is not explicite.

Thm (Bruveris, Vialard '14)

If $V = H^k(\mathbb{R}^d, \mathbb{R}^d)$ with $k > \frac{d}{2} + 1$ then $G_V = Diff^k(\mathbb{R}^d)$ and G_V is also ***geodesically complete***

Finite dimensional approximations

- Key induction property for homogeneous shape spaces under the same group G

Let $G \times M' \to M'$ and $G \times M \to M$ be defining two homegeneous shape spaces and assume that $\pi : M' \to M$ is a onto mapping such that

$$\pi(gm') = g\pi(m')\,.$$

Then

$$d_M(m_0, m_1) = d_{M'}(\pi^{-1}(m_0), \pi^{-1}(m_1))\,.$$

Consequence: if $M_n = \lim \uparrow M_\infty$ we can approximate geodesics on M_∞ from geodesic on the finite dimensional approximations M_n.

Basis for **landmarks based approximations** of many shape spaces of submanifolds.

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Shooting on homogeneous shape space

For $(q, v) \mapsto \xi_q(v)$ (infinitesimal transport) we end up with an optimal control problem

$$\begin{cases} \min \int_0^1 (Lv|v)dt \\ \\ \text{subject to} \\ q(0), q(1) \text{ fixed}, \ \dot{q} = \xi_q(v) \end{cases}$$

The solution can be written in hamiltonian form: with

$$H(q, p, v) = (p|\xi_q(v)) - \frac{1}{2}(Lv|v)\,.$$

Reduction from PMP:

$$H(q, p) = \frac{1}{2}(K\xi_q^*(p)|\xi_q^*(p))$$

Smooth as soon as $(q, v) \mapsto \xi_q(v)$ **is smooth**. No metric derivative !
(*Arguillière, Trelat, T., Younes'14*)

Why shooting is good

Let consider a generic optimization problem arising from shooting:
Let $z = (q, p)^T$, $F = \left(\ \partial_p H, -\partial_q H \ \right)^T$ (R and U smooth enough)

$$\begin{cases} \min_{z(0)} R(z(0)) + U(z(1)) \\ \text{subject to} \\ Cz(0) = 0, \dot{z} = F(z) \end{cases}$$

Gradient scheme through a *forward-backward* algorithm:

- Given $z_n(0)$, **shoot forward** ($\dot{z} = F(z)$) to get $z_n(1)$.
- Set $\eta_n(1) + dU(z_n(1)) = 0$ and **integrate backward** the *adjoint* evolution until time 0

$$\dot{\eta} = -dF^*(z_n)\eta$$

The gradient descent direction D_n is given as

$$D_n = C^*\lambda - \nabla R(z_n(0)) + \eta_n(0)$$

An extremely usefull remark (S. Arguillère '14)

If $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, we have

$$F = J\nabla H$$

so that

$$dF = J\, d(\nabla H) = J\mathrm{Hess}(H)$$

Since the **hessian is symmetric** we get

$$dF^* = JdFJ$$

Hence

$$dF(z)^*\eta = J\frac{d}{d\varepsilon}(F(z+\varepsilon\eta))_{|\varepsilon=0}J$$

so that we get the backward evolution **at the same cost** than the forward via a finite difference scheme.

Shooting the painted bunny (fixed template)

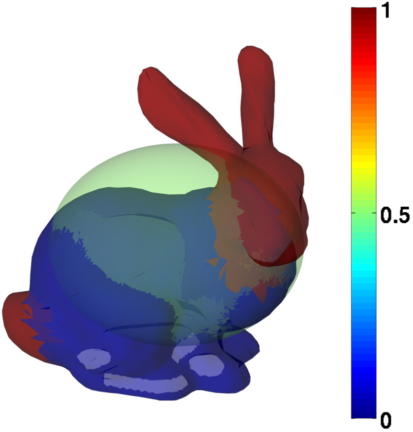
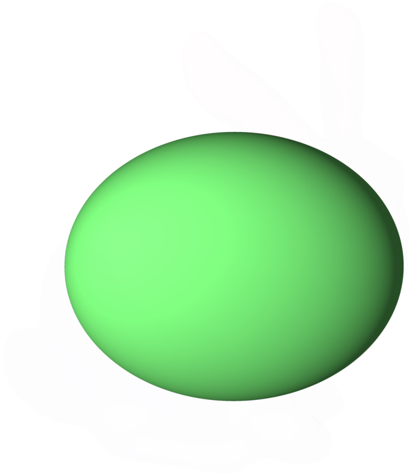


Figure: Shooting from fixed template (painted bunny

(*Charlier, Charon, T.'14*)

Shooting the bunny...



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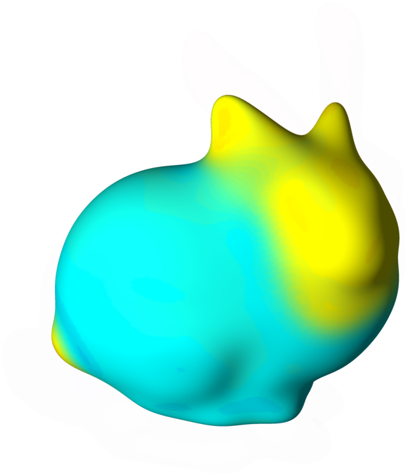

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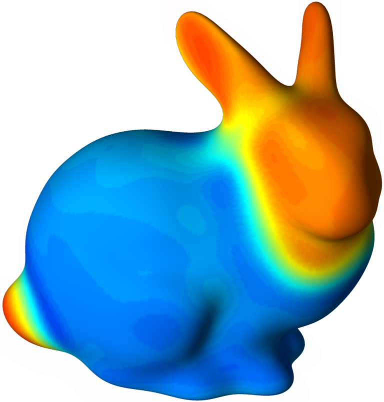

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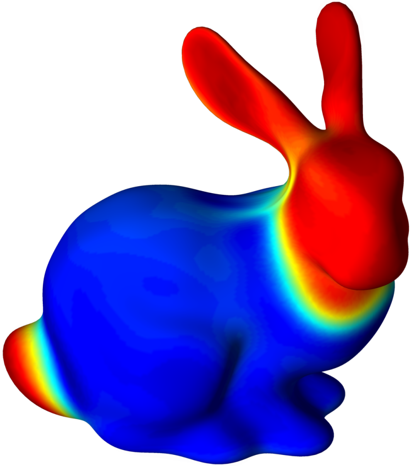

Thank You.