Geodesic shooting on shape spaces

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Outline

Riemannian manifolds (finite dimensional)

Spaces spaces (intrinsic metrics)

Diffeomorphic transport and homogeneous shape spaces

Geodesic shooting on homogeneous shape spaces
Riemannian geometry

The classical apparatus of (finite dimensional) *riemannian geometry* starts with the definition of a **metric** $\langle \ , \rangle_m$ on the tangent bundle.

**Geodesics and energy**

Find the path $t \rightarrow \gamma(t)$ from $m_0$ to $m_1$ minimizing the energy

$$I(\gamma) \triangleq \int_0^1 \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)} dt$$

Critical paths from $I$ are **geodesics**

![Diagram of a path γ(t) from m_0 to m_1](image)
Geodesic equation

\[ \frac{dl}{ds}(\gamma) = -\int_0^T \langle \frac{D}{dt}\dot{\gamma}, \frac{\partial}{\partial s}\gamma \rangle \gamma(t) dt \]

Figure: Variations around \( \gamma(t) \)

\[ \delta l \equiv 0 \text{ for } \frac{D}{dt}\dot{\gamma} \equiv 0 \]

where \( \frac{D}{dt} = \nabla_{\dot{\gamma}} \) is the covariant derivative along \( \gamma \)

Second order EDO given \( \gamma(0), \dot{\gamma}(0) \).
Exponential Mapping and Geodesic Shooting

This leads to the definition of the exponential mapping

$$\text{Exp}_{\gamma(0)} : T_{\gamma(0)}M \to M.$$ 

Starts at $m_0 = \gamma(0)$, chooses the direction $\gamma'(0) \in: T_{\gamma(0)}M$ and shoots along the geodesic to $m_1 = \gamma(1)$.

Figure: Exponential mapping and normal coordinates

Key component of many interesting problems: Generative models, Karcher means, parallel transport via Jacobi fields, etc.
Lagrangian Point of View

(In local coordinates)

► Constrained minimization problem

$$\int_0^1 L(q(t), \dot{q}(t)) \, dt$$

with Lagrangian $L(q, \dot{q}) = \frac{1}{2} |\dot{q}|_q^2 = \frac{1}{2} (L_q \dot{q} |\dot{q})$ and $(q_0, q_1)$ fixed

$L_q$ codes the metric. $L_q$ symmetric positive definite.

► Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0$$
From Lagrangian to Hamiltonian Variables

- Change \((q, \dot{q})\) (position, velocity) \(\rightarrow\) \((q, p)\) (position, momentum) with
  \[ p = \frac{\partial L}{\partial \dot{q}} = L_q \dot{q} \]

- Euler-Lagrange equation is equivalent to the **Hamiltonian** equations:
  \[
  \begin{align*}
  \dot{q} &= \frac{\partial H}{\partial p}(q, p) \\
  \dot{p} &= -\frac{\partial H}{\partial q}(q, p)
  \end{align*}
  \]
  where (Pontryagin Maximum Principle)

  \[ H(q, p) = \max_u (p|u) - L(q, u) = \frac{1}{2}(K_q p|p) \]

  \(K_q = L_q^{-1}\) define the co-metric.

Note: \(\partial_q H\) induces the derivative of \(K_q\) with respect to \(q\).
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The *ideal* mathematical setting: A smart space $Q$ of smooth mappings from a smooth manifold $S$ to $\mathbb{R}^d$.

Basic spaces are $\text{Emb}(S, \mathbb{R}^d)$, $\text{Imm}(S, \mathbb{R}^d)$ the space of smooth (say $C^\infty$) embeddings or immersions from $S$ to $\mathbb{R}^d$. May introduce a finite regularity $k \in \mathbb{N}^*$ and speak about $\text{Emb}^k(S, \mathbb{R}^d)$ and $\text{Imm}^k(S, \mathbb{R}^d)$.

$S = S^1$ for close curves, $S = S^2$ for close surfaces homeomorphic to the sphere

Nice since open subset of $C^\infty(S, \mathbb{R})$. For $k > 0$, open subset of a Banach space.
Case of curves: $S^1$ is the unit circle.

- $L^2$ metric: $h, h' \in T_q C^\infty(S, \mathbb{R}^d)$
  $$\langle h, h' \rangle_q = \int \langle h, h' \rangle |\partial_\theta q| d\theta = \int_{S^1} \langle h, h' \rangle ds.$$ 

- Extensions in Michor and Mumford (06)

- $H^1$ type metric:
  $$\langle h, h' \rangle_q = \int_{S^1} \langle (D_s h)^\perp, (D_s h')^\perp \rangle + b^2 \langle (D_s h)^\top, (D_s h')^\top \rangle ds$$

  where $D_s = \partial_\theta / |\partial_\theta q|$ 

  - Younes’s elastic metric (Younes ’98, $b = 1$, $d = 2$), Joshi Klassen Srivastava Jermyn ‘07 for $b = 1/2$ and $d \geq 2$ (SRVT trick).
Parametrization invariance: \( \psi \in \text{Diff}(S) \)

\[
\langle h \circ \psi, h' \circ \psi \rangle_{q \circ \psi} = \langle h, h' \rangle_q.
\]

- Sobolev metrics (Michor Mumford '07; Charpiat Keriven Faugeras '07; Sundaramoorthi Yezzi Mennuci '07): \( a_0 > 0 \), \( a_n > 0 \)

\[
\langle h, h' \rangle_q = \int_{S^1} \sum_{i=0}^{n} a_i \langle D_s^i h, D_s h' \rangle ds.
\]

Again, parameterization invariant metric.

- Extension for surfaces (\( \dim(S) \geq 2 \)) in Bauer Harms Michor '11.
Summary and questions

- Many possible metrics on the preshape spaces $Q$ (how to choose)
- Ends up with a smooth parametrization invariant metric on a smooth preshape space $Q$ and a riemannian geodesic distance.

Questions: Minimal: Local existence of geodesic equations and smoothness for smooth data? More

1. Existence of global solution (in time) of the geodesic equation (geodesically complete metric space)?
2. Existence of a minimising geodesic between any two points (geodesic metric space)?
3. Completeness of the space for the geodesic distance (complete metric space)?

1-2-3 equivalents on finite dimensional riemannian manifold (Hopf-Rinow thm)
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Few answers

- (Local solution): Basically, for Sobolev norm of order $n$ greater than 1, local existence of solutions of the geodesic equation if the initial data has enough regularity (*Bauer Harms Michor ’11*):

  $$k > \frac{\dim(S)}{2} + 2n + 1$$

- (Geodesic completeness): Global existence has been proved recently for $S = S^1$, $d = 2$ (planar shapes) and $n = 2$ (*Bruveris Michor Mumford ’14*). Wrong for the order 1 Sobolev metric. Mostly unkown for the other cases.

- (Geodesic metric spaces): Widely open

- (Complete metric space): No for smooth mappings (weak metric). Seems to be open for $\text{Imm}^k(S, \mathbb{R}^d)$ or $\text{Emb}^k(S, \mathbb{R}^d)$ and order $k$ Sobolev metric.
Why there is almost no free lunch

Back to the Hamiltonian point of view. The metric can be written \((L_q h| h)\) with \(L_h\) an elliptic symmetric definite differential operator.

\[
H(q, p) = \frac{1}{2}(K_q p|q)
\]

where \(K_q = L_q^{-1}\) is a pseudo-differential operator with a really intricate dependency with the pre-shape \(q\).
Towards shape shapes: removing parametrisation

**Diff(S)** as a nuisance parameter

- **Diff(S):** the diffeomorphism group on \( S \) (reparametrization).
- **Canonical shape spaces:** \( \text{Emb}(S, \mathbb{R}^d)/\text{Diff}(S) \) or \( \text{Imm}(S, \mathbb{R}^d)/\text{Diff}(S) \)

\[
[q] = \{ q \circ \psi \mid \psi \in \text{Diff}(S) \}
\]

- **Structure of manifold** for \( \text{Emb}(S, \mathbb{R}^d)/\text{Diff}(S) \) and \( \text{Imm}(S, \mathbb{R}^d)/\text{Diff}(S) \) (orbifold)
- **Induced geodesic distance**

\[
d_{\mathcal{Q}/\text{Diff}(S)}([q_0], [q_1]) = \inf \{ d_{\mathcal{Q}}(q_0, q_1 \circ \psi) \mid \psi \in \text{Diff}(S) \} 
\]
Questions: Given to two curves $q_0$ and $q_{\text{targ}}$ representing two shapes $[q_0]$ and $[q_{\text{targ}}]$

- Existence of an horizontal geodesic path $t \mapsto q_t \in Q$ emanating from $q_0$ and of a reparametrisation path $t \mapsto \psi_t \in \text{Diff}(S)$ such that $q_{\text{targ}} = q_1 \circ \psi_1$?

No available shooting algorithms for parametrized curves or surfaces, only mainly path straightening algorithms or DP algorithms that alternate between $q$ and $\psi$.

Usually, no guarantee of existence of an optimal diffeomorphic parametrisation $\psi_1$ (T. Younes ’97).
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Shape spaces as homogeneous spaces

Idea #1:
D’Arcy Thomspson and Grenander. Put the emphasis on the left action of the group of diffeomorphisms on the embedding space $\mathbb{R}^d$ and consider homogeneous spaces $M = G.m_0$:

$$G \times M \rightarrow M$$

Diffeomorphisms can act on almost everything (changes of coordinates)!

Idea #2:
Put the metric on the group $G$ (right invariance). More simple. Just need to specify the metric at the identity.
Idea #3:
Build the metric on $M$ from the metric on $G$:

1. If $G$ has a $G$ (right)-equivariant metric:

$$d_G(g_0 g, g_0 g') = d_G(g, g')$$ for any $g_0 \in G$

then $M$ inherits a quotient metric

$$d_M(m_0, m_1) = \inf \{ d_G(Id, g) \mid gm_0 = m_1 \in G \}$$

2. The geodesic on $Gm_0$ can be lifted to a geodesic in $G$ (horizontal lift).
Construction of right-invariant metrics

Start from a Hilbert space $V \hookrightarrow C^1_0(\mathbb{R}^d, \mathbb{R}^d)$.

1. Integrate time dependent vector fields $v(.) = (v(t))_{t \in [0,1]}$ :

$$\dot{g} = v \circ g, \quad g(0) = \text{Id}.$$ 

2. Note $g^v(.)$ the solution and

$$G_V \doteq \{ g^v(1) \mid \int_0^1 |v(t)|^2_V dt < \infty \}.$$

$$d_{G_V}(g_0, g_1) \doteq \left( \inf \{ \int_0^1 |v(t)|^2_V dt < \infty \mid g_1 = g^v(1) \circ g_0 \} \right)^{1/2}$$
Basic properties

Thm (T.)

If $V \rightarrow C^1_0(\mathbb{R}^d, \mathbb{R}^d)$ then

1. $G_V$ is a group of $C^1$ diffeomorphisms on $\mathbb{R}^d$.
2. $G_V$ is a complete metric space for $d_G$
3. we have existence of a minimizing geodesic between any two
group elements $g_0$ and $g_1$ (geodesic metric space)

Note: $G_V$ is parametrized by $V$ which is not a Lie algebra. Usually $G_V$ and $d_G$ is not explicite.

Thm (Bruveris, Vialard ’14)

If $V = H^k(\mathbb{R}^d, \mathbb{R}^d)$ with $k > \frac{d}{2} + 1$ then $G_V = \text{Diff}^k(\mathbb{R}^d)$ and $G_V$ is also
geodesically complete
Finite dimensional approximations

- Key induction property for homogeneous shape spaces under the same group $G$

Let $G \times M' \to M'$ and $G \times M \to M$ be defining two homogeneous shape spaces and assume that $\pi : M' \to M$ is a onto mapping such that

$$\pi(gm') = g\pi(m').$$

Then

$$d_M(m_0, m_1) = d_{M'}(\pi^{-1}(m_0), \pi^{-1}(m_1)).$$

Consequence: if $M_n = \lim \uparrow M_\infty$ we can approximate geodesics on $M_\infty$ from geodesic on the finite dimensional approximations $M_n$.

Basis for landmarks based approximations of many shape spaces of submanifolds.
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Shooting on homogeneous shape space

For \((q, v) \mapsto \xi_q(v)\) (infinitesimal transport) we end up with an optimal control problem

\[
\begin{align*}
\min & \int_0^1 (Lv|v)dt \\
\text{subject to} & \\
q(0), q(1) & \text{fixed, } \dot{q} = \xi_q(v)
\end{align*}
\]

The solution can be written in hamiltonian form: with

\[
H(q, p, v) = (p|\xi_q(v)) - \frac{1}{2}(Lv|v).
\]

Reduction from PMP:

\[
H(q, p) = \frac{1}{2}(K\xi_q^*(p)|\xi_q^*(p))
\]

**Smooth as soon as** \((q, v) \mapsto \xi_q(v)\) **is smooth.** No metric derivative!  
(Arguillière, Trelat, T., Younes’14)
Why shooting is good

Let consider a generic optimization problem arising from shooting:
Let $z = (q, p)^T$, $F = (\partial_p H, -\partial_q H)^T$ ($R$ and $U$ smooth enough)

\[
\begin{align*}
\min_{z(0)} & \quad R(z(0)) + U(z(1)) \\
\text{subject to} & \\
Cz(0) &= 0, \quad \dot{z} = F(z)
\end{align*}
\]

Gradient scheme through a forward-backward algorithm:

- Given $z_n(0)$, **shoot forward** ($\dot{z} = F(z)$) to get $z_n(1)$.
- Set $\eta_n(1) + dU(z_n(1)) = 0$ and **integrate backward** the adjoint evolution until time 0

\[
\dot{\eta} = -dF^*(z_n)\eta
\]

The gradient descent direction $D_n$ is given as

\[
D_n = C^*\lambda - \nabla R(z_n(0)) + \eta_n(0)
\]
An extremely usefull remark (S. Arguillère ’14)

If $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, we have

$$F = J \nabla H$$

so that

$$dF = J d(\nabla H) = J \text{Hess}(H)$$

Since the hessian is symmetric we get

$$dF^* = J dF$$

Hence

$$dF(z)^* \eta = J \frac{d}{d\varepsilon}(F(z + \varepsilon \eta))|_{\varepsilon=0} J$$

so that we get the backward evolution at the same cost than the forward via a finite difference scheme.
Shooting the painted bunny (fixed template)

Figure: Shooting from fixed template (painted bunny)

(Charlier, Charon, T.'14)
Shooting the bunny...
Shooting the bunny...
Shooting the bunny...
Shooting the bunny...
Thank You.