Variational denoising for manifold-valued data

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Overview

An algorithm for TV minimization for manifold-valued data (joint work with L. Demaret and M.Storath)

Second order TV type functionals for $\mathbb{S}^1$-valued data (joint work with R. Bergmann, F. Laus, G. Steidl)

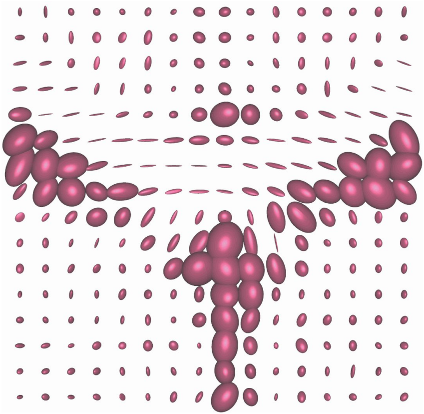
Potts and Blake-Zisserman functionals for manifold-valued signals with a few jumps (joint work with L. Demaret and M.Storath)

Manifold-valued data in DTI

- In diffusion tensor imaging (DTI) (Basser et al. '94) the data are positive(-definite) matrices.
- It is reasonable (cf. Pennec et al. '2004) to equip Pos_n with the Riemannian metric

$$g_P(A, B) = \mathrm{trace}(P^{-\frac{1}{2}} A P^{-1} B P^{-\frac{1}{2}}),$$

P positive and A, B symmetric.



Positive Matrices visualized as ellipsoids.

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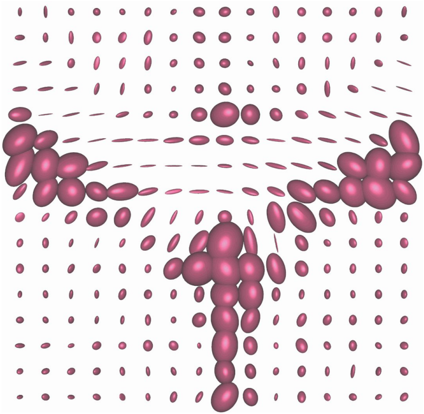
P positive and A, B symmetric.

- Pos_n with the metric g_P is a Cartan Hadamard manifold (complete, nonpositive sectional curvature, simply connected).



Positive Matrices visualized as ellipsoids.

- log and exp can be computed explicitly by

$$\log_P Q = P^{\frac{1}{2}} \log(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}}) P^{\frac{1}{2}}, \qquad \exp_P A = P^{\frac{1}{2}} \exp(P^{-\frac{1}{2}} A P^{-\frac{1}{2}}) P^{\frac{1}{2}}.$$

TV functionals for manifold-valued data

We consider the variational denoising problem given by the (discrete, anisotropic, bivariate) functionals

$$F_\alpha(u) = \sum_{i,j} \mathrm{dist}(u_i, f_i)^p + \alpha \sum_{i,j} \mathrm{dist}(u_{ij}, u_{i-1,j})^q + \alpha \sum_{i,j} \mathrm{dist}(u_{ij}, u_{i,j-1})^q,$$

with data f and $p, q \geq 1$.

- Choosing q=1 corresponds to (anisotropic) TV minimization/ROF model in Lagrange form (Rudin, Osher, Fatemi ’90).

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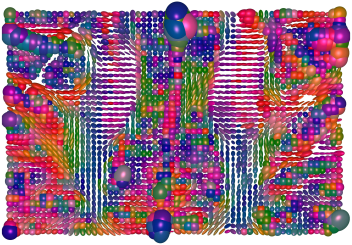
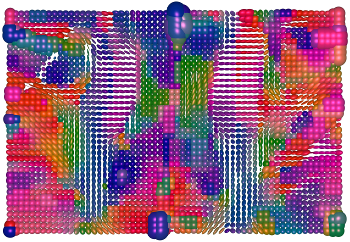
- Choose the Riemannian distance dist to obtain the corresponding functionals for manifold-valued data.
- Increase anisotropy by additionally considering diagonals, knight moves, ...

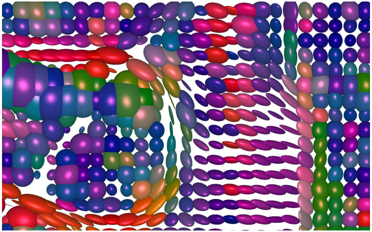
TV denoising on real DTI data (Camino project, Cook et. al. '06)

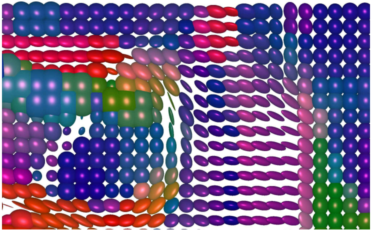




Real data

Our method for $\ell^2 - TV$
$\alpha = 0.11$.





Minimization algorithms - TV problem

- **Idea:** Write (for simplicity univariate, multivarite analogous):

$$F(u) = \gamma \sum_i \mathrm{dist}(u_i, u_{i-1})^q + \sum_j \mathrm{dist}(u_j, f_j)^p = \sum_i F_i(u) + G(u),$$

$$\text{where} \quad F_i(u) = \gamma \, \mathrm{dist}(u_i, u_{i-1})^q, \quad G = \sum_j \mathrm{dist}(u_j, f_j)^p.$$

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- Apply the cyclic proximal point algorithm (Bacak, Bertsekas) : Iterate the proximal mappings (Moreau) of G and F_i, $i = l, \ldots, r$,

$$\operatorname{prox}_{\lambda F_i}(u) = \arg\min_v \tfrac{1}{2}\operatorname{dist}(u, v)^2 + \lambda F_i(v).$$

- **Central Point:** The proximal mappings of F_i, G can be computed explicitely (next slide).

Minimization algorithms - TV problem

Minimize $F(u) = \sum_i F_i(u) + G(u)$,

$$F_i(u) = \gamma\,\mathrm{dist}(u_i, u_{i-1})^q, \quad G = \sum_j \mathrm{dist}(u_j, f_j)^p.$$

- The proximal mapping of G is explicitly given by

$$\mathrm{prox}_{\lambda G}(u)_i = [u_i, f_i]_t, \quad t = \begin{cases} \frac{2\lambda}{(1+2\lambda)}\mathrm{dist}(u_i, f_i) & \text{for p= 2,} \\ \min(\lambda, \mathrm{dist}(u_i, f_i)) & \text{for p= 1.} \end{cases}$$

("Soft thresholding" for $p = 1$.)

- The proximal mapping of F_i is explicitly given by (Demaret, Storath, W.)

$$\mathrm{prox}_{\lambda F_i}(u)_j = \begin{cases} u_j & \text{if } j \neq i, i-1, \\ [u_i, u_{i-1}]_t & \text{if } j = i, \\ [u_{i-1}, u_i]_t & \text{if } j = i-1, \end{cases}$$

$t = \frac{\gamma\lambda}{(2+2\gamma\lambda)}\mathrm{dist}(u_i, u_{i-1})$ for q=2, $t = \min(\lambda\gamma, \frac{1}{2}\mathrm{dist}(u_i, u_{i-1}))$ for q=1.

Minimization algorithms - TV problem

Minimize $F(u) = \sum_i F_i(u) + \sum_i G_i(u)$,

$$F_i(u) = \gamma \operatorname{dist}(u_i, u_{i-1})^q, \quad G_i = \operatorname{dist}(u_i, f_i)^p.$$

- **A parallel proximal point algorithm**: Calculate the proximal mappings of F_i, G_i at $u^{(k)}$

$$u_i^{(k+1)} = \operatorname{prox}_{\lambda F_i}(u^{(k)}), \quad u_{n+i}^{(k+1)} = \operatorname{prox}_{\lambda G_i}(u^{(k)}),$$

and then average them using **intrinsic means** (Cartan, Frechet, Karcher, ...)

$$u^{(k+1)} = \arg\min_u \sum_i \operatorname{dist}(u, u_i^{(k+1)})^2.$$

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$$v = u^{(k+1)} = \arg\min_u \sum_i \mathrm{dist}(u, u_i^{(k+1)})^2.$$

- To compute the minimizer, we use the gradient descent (Karcher)

$$v_{\mathrm{new}} = \exp_{v_{\mathrm{old}}}\left(\frac{1}{N} \sum_{i=1}^N \log_{v_{\mathrm{old}}} u_i^{(k+1)}\right).$$

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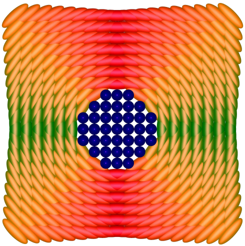
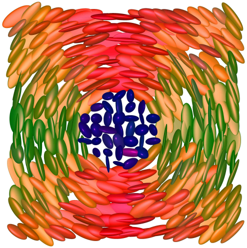
- **Fast Variant:** Approximate the mean by iterated geodesic averages.

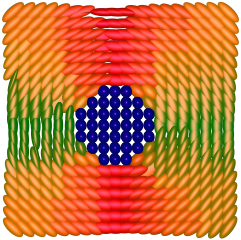
Synthetic DTI example



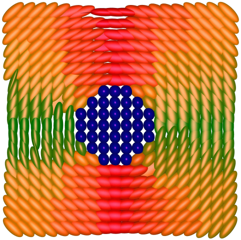
Synthetic DT image



Rician noise, $\sigma = 90$.



ℓ^2-TV (our cyclic PPA)



ℓ^2-TV (our parallel PPA)

Analytic Results

Theorem (Demaret, Storath, W.)

In a Cartan-Hadamard manifold (complete, simply connected, nonpositive sectional curvature) the proposed algorithms (cyclic, parallel and the parallel variant with approximative mean computation) for L^p-TV minimization converge towards a global minimizer.

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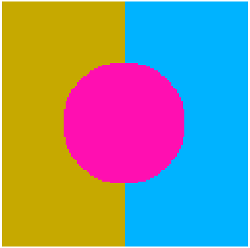
In a Cartan-Hadamard manifold (complete, simply connected, nonpositive sectional curvature) the proposed algorithms (cyclic, parallel and the parallel variant with approximative mean computation) for L^p-TV minimization converge towards a global minimizer.

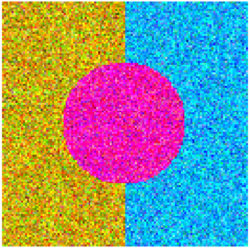
Sceleton of proof:

- Proof that in a connected, complete Riemannian manifold, the proximal mappings of the first differences and the distances are given by the formulas derived above.
- For the cyclic PPA apply the convergence result of Bacak (Bacak '14).
- For the parallel PPAs base on techniques used in (Bacak '14) and find suitable modifications.

Denoising on the LCh color model ($\mathbb{S}^1 \times \mathbb{R}^2$).

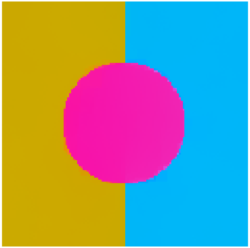


Synthetic image



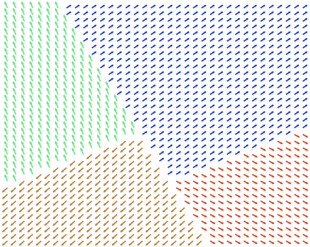
Gaussian noise (PSNR: 15.64).

ℓ^2-TV on RGB (PSNR:23.92)

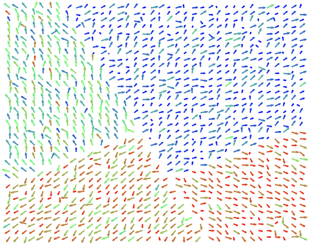


ℓ^2-TV on LCh (PSNR:32.19)

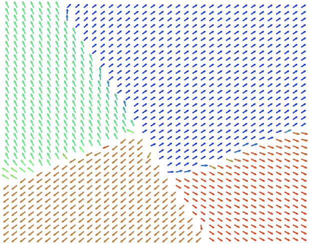
Denoising $\mathbb{S}^2$ data.



Original

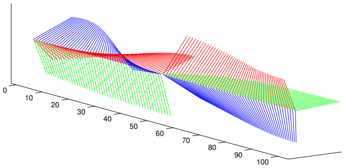


Von Mises-Fisher noise ($\kappa = 12.7$)

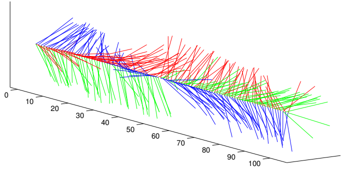


ℓ^1-TV regularization

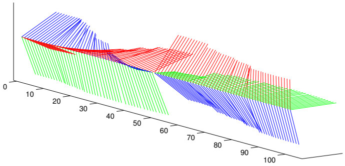
Denoising SO_3 data.



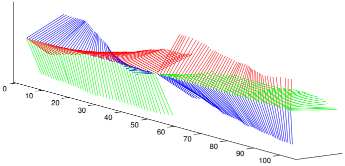
Synthetic signal



Fisher noise ($\kappa = 75.$)

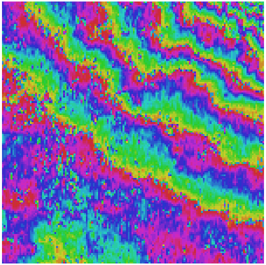


ℓ^2-TV

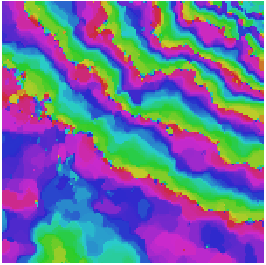


ℓ^2-Huber

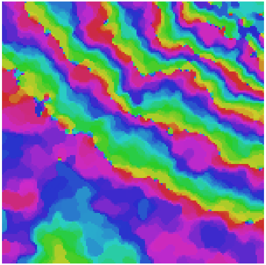
Denoising inSAR data



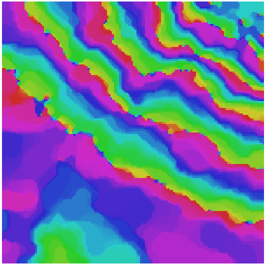
Real data



L^2-TV denoising



L^1-TV denoising



TV with Huber data term

Part II:

Second order TV type functionals for $\mathbb{S}^1$-valued data

(joint work with R. Bergmann, F. Laus, G. Steidl)

Second order TV for S^1 valued data

- Second order TV type functional for real-valued data:

$$F(u) = \|u - f\|_2^2 + \alpha\|\nabla_1 u\|_1 + \beta\|\nabla_2 u\|_1.$$

Here,

$$\nabla_2 u(i) = u(i-1) - 2u(i) + u(i+1).$$

- **Question:** What are second differences for S^1 valued data?

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- **Idea:** Translate

$$\nabla_2 u(i) = (u(i-1) - u(i)) + (u(i-1) - u(i))$$

  to the manifold setting:

$$\nabla_2 u(i) = \exp_{u(i)}^{-1} u(i-1) + \exp_{u(i)}^{-1} u(i+1).$$

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- **Problem:** These second differences are not continuous in u_{i-1}, u_i, u_{i+1}.

Second order TV for S^1 valued data

- **Alternative**: View $u_i \in]-\pi, \pi]$ as real-valued data and define the absolute cyclic difference

$$d_2(f_{i-1}, f_i, f_{i+1}) = \min_{k,l,m=-1,0,1} |\nabla_2(f_{i-1} + k2\pi, f_i + l2\pi, f_{i+1} + m2\pi)|$$

  These differences are continuous in f_{i-1}, f_i, f_{i+1}.
- Equivalent: Consider all liftings and take the minimal difference on the lifted $\mathbb{R}$-valued data.
- For nearby f_{i-1}, f_i, f_{i+1} the manifold and the lifting definition agree.

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These differences are continuous in f_{i-1}, f_i, f_{i+1}.

- Equivalent: Consider all liftings and take the minimal difference on the lifted $\mathbb{R}$-valued data.
- For nearby f_{i-1}, f_i, f_{i+1} the manifold and the lifting definition agree.
- The proximal mappings for d_2 can be computed explicitely (Bergmann, Laus, Steidl, W. '14): for $w = (1, -2, 1)^T$, and $|\langle f, w\rangle| < \pi$,

$$\operatorname{prox}_{\lambda d_2}(f) = (f - swm)_{2\pi}, \quad m = \min\left(\lambda, \frac{\langle f,w\rangle}{\|w\|_2^2}\right), \quad s = \operatorname{sign}\langle f, w\rangle.$$

- All ingredients for the cyclic proximal point algorithm are available.

Convergence of the cyclic proximal point algorithms

Theorem (Bergmann, Laus, Steidl, W., 2014)

For data f with nearby data items and small enough parameters α, β, the cyclic proximal point algorithm for second order TV type minimization converges to a minimizer.

- What nearby means and α, β can be quantified.

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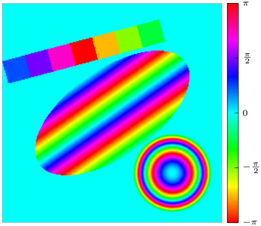
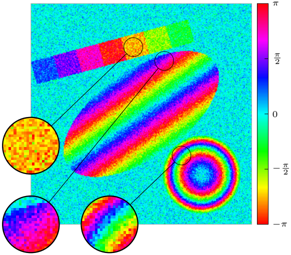
Idea of proof:

- Lift the setting to the covering space $\mathbb{R}$.
- For $\mathbb{R}$-valued data we have convergence and the distance of the iterates can be estimated basing on (Bacak, Bertsekas).
- Lifting commutes with the proximal mappings and all other relevant operations for the considered data.
- Conclude nearness for $\mathbb{S}^1$ data and derive convergence.

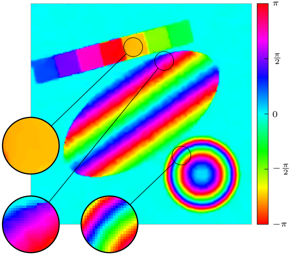
Second order TV minimization - synthetic example.



Original



Noisy



Second order TV.

Denoising the H channel in HSV space.

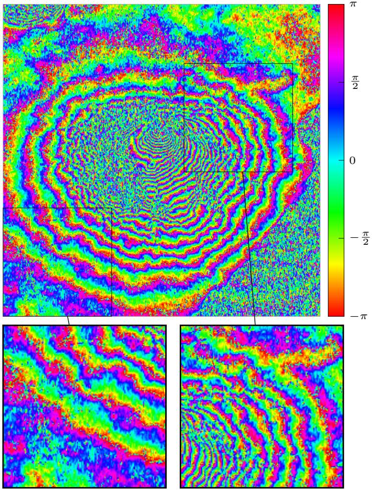
Image

Noisy hue

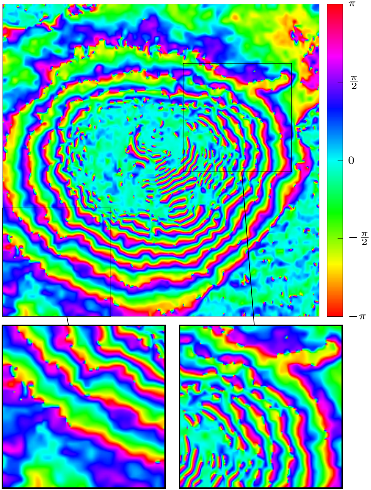
hue denoising on $\mathbb{R}$

hue denoising on $\mathbb{S}^1$

Second order TV for real SAR data of Mt. Vesuvius.



Original



Second order TV denoising.

Part III:

Potts and Blake-Zisserman functionals for manifold-valued signals with a few jumps

(joint work with L. Demaret and M.Storath)

Potts and Blake-Zisserman functionals for manifold-valued data

Define (univariate) Potts functionals P_γ for manifold-valued data,

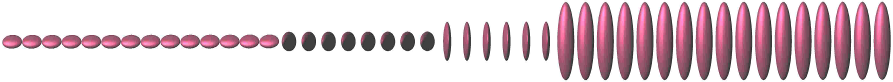
$$P_\gamma(u) = \gamma \,\#\{i : u_i \neq u_{i-1}\} + \sum\nolimits_i \mathrm{dist}(u_i, f_i)^p,$$

and Blake-Zisserman functionals B_γ for manifold-valued data,

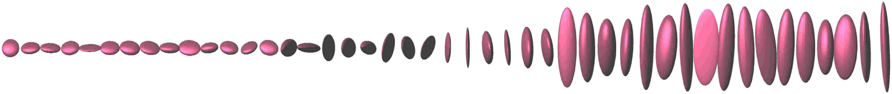
$$B_\gamma(u) = \gamma \,\sum\nolimits_i \min(s^q, \mathrm{dist}(u_i, u_{i-1})^q) + \sum\nolimits_i \mathrm{dist}(u_i, f_i)^p.$$

Example: L^2-Potts minimization for DTI data.

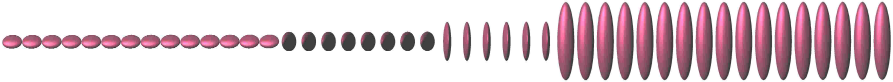
Original (synthetic) signal:



Noisy data (Rician noise with $\sigma = 60$):

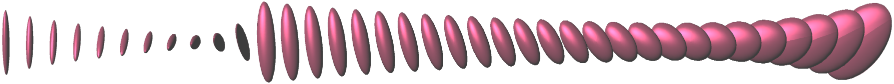


L^2-Potts reconstruction:

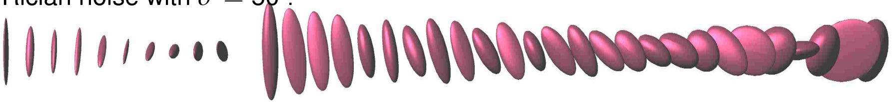


Example: Blake-Zisserman vs. Potts.

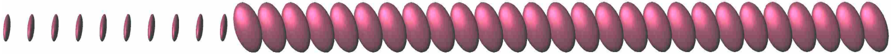
Original (synthetic) signal:



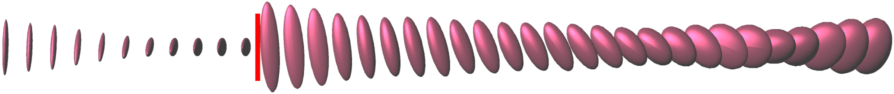
Rician noise with $\sigma = 50$:



L^2-Potts reconstruction:



L^2-Blake-Zisserman reconstruction:



Minimization algorithm - Potts problem

$$P_\gamma(u) = \gamma \, \#\{i : u_i \neq u_{i-1}\} + \sum_{i=1}^{n} \mathrm{dist}(u_i, f_i)^p \to \min,$$

- Algorithm based on dynamic programming.
- Most time consuming: For each subinterval $[l, r]$ calculate

$$u = \arg\min \sum_{i=l}^{r} \mathrm{dist}(u, f_i)^p,$$

(p=2: Riemannian center of mass; p=1: Riemannian median.)

We use (sub-)gradient descent; e.g., for $p = 1$,

$$u^{k+1} = \exp_{u^k}\left(\tau_k \sum_{i=l}^{r} \frac{\log_{u^k} f_i}{\|\log_{u^k} f_i\|}\right).$$

Converges for $p = 1$ when $\tau \in \ell^2 \setminus \ell^1$ (Arnaudon et al. '11).

Minimization algorithm - Blake-Zisserman problem

$$B_\gamma(u) = \gamma \sum\nolimits_i \min(s^q, \mathrm{dist}(u_i, u_{i-1})^q) + \sum\nolimits_j \mathrm{dist}(u_j, f_j)^p \to \min.$$

- Algorithm based on dynamic programming.
- For each subinterval $[l, r]$ calculate the minimizer of

$$F(u) = \gamma \sum\nolimits_i \mathrm{dist}(u_i, u_{i-1})^q + \sum\nolimits_j \mathrm{dist}(u_j, f_j)^p$$

- This is a *TV* minimization problem (or, more general, ℓ^q variation minimization) for manifold data which can be solved by the developed methods.

Minimization algorithms

Theorem (Demaret, Storath, W.)

Let $p, q \geq 1$. In a Cartan-Hadamard manifold, our algorithm for the minimization of the (univariate) Potts functionals P_γ produces a minimizer.

Theorem (Demaret, Storath, W.)

Let $p, q \geq 1$. In a Hadamard space, our algorithm for the minimization of the (inivariate) Blake-Zisserman functionals B_γ produces a minimizer.

Minimization algorithms

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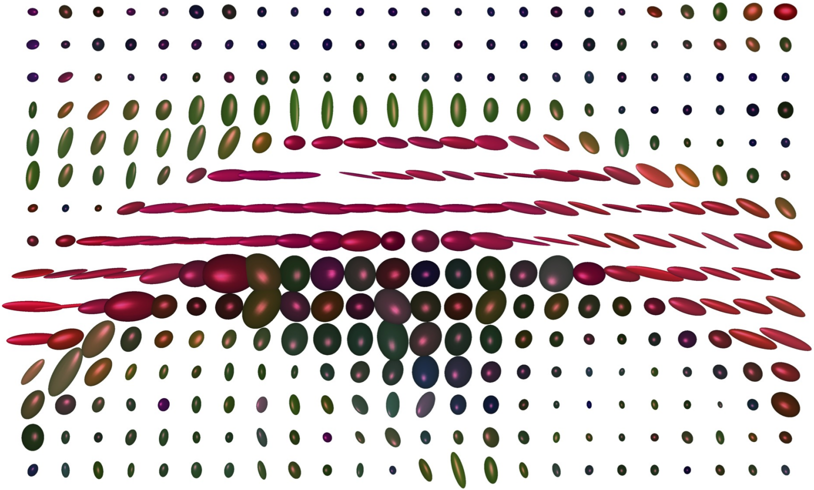
Theorem (Demaret, Storath, W.)

Let $p, q \geq 1$. In a Hadamard space, our algorithm for the minimization of the (inivariate) Blake-Zisserman functionals B_γ produces a minimizer.

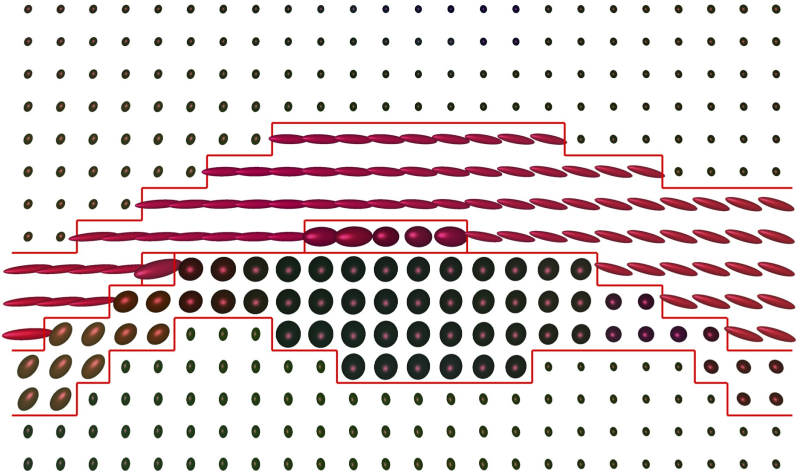
- For multivariate data, the Potts and the Blake-Zisserman problem are NP hard.
- In this case, we use a splitting approach (cf. W., Demaret, Storath '14).

Segmentation: real data from the Camino project (Cook et al. '06)



Segmentation: real data from the Camino project (Cook et al. ’06)

Edge between neighbouring $P, Q \iff \mathrm{dist}(P, Q) \geq s$ (s B.-Z. parameter).



Blake-Zisserman regularization ($p, q = 1$) with $s = 0.67$, $\gamma = 4.3$.

Summary

- We have derived algorithms for TV minimization for manifolds.
- We have shown convergence to a minimizer for Hadamard manifolds.
- We have seen the potential in various applications.
- We have derived an algorithm for second order TV type functionals for S^1 data.
- We have obtained convergence for nearby neighboring data and shown applications.
- We have obtained algorithms for Potts and Blake-Zisserman problems for manifold valued data.
- We have seen a segmentation of a real corpus callosum.

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