Lower Semi-Continuity of Non-Local Functionals

Peter Elbau

Johann Radon Institute for Computational and Applied Mathematics

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Outline

Local Functionals

Let throughout the talk be $X \subset \mathbb{R}^m$, $m \in \mathbb{N}$, a bounded, open set. Let further $u_0 : X \to \mathbb{R}$ be a given intensity image.
A standard way to denoise the image u_0 is to replace it by a minimising point of a (local) energy functional of the form

$$\mathcal{E}(u) = \int_X e(x, u(x), \nabla u(x))\,\mathrm{d}x$$

on some Sobolev space. A common choice is

$$\mathcal{E} : W^{1,q}(X) \to \mathbb{R} \cup \{\infty\}, \quad \mathcal{E}(u) = \int_X (u - u_0)^2\,\mathrm{d}x + \alpha \int_X |\nabla u|^q\,\mathrm{d}x$$

for some regularisation parameter $\alpha \in (0, \infty)$ and some $q \in (1, \infty)$.
For $q = 1$ this generalises to the total variation functional

$$\mathcal{E} : BV(X) \to \mathbb{R} \cup \{\infty\}, \quad \mathcal{E}(u) = \int_X (u - u_0)^2\,\mathrm{d}x + \alpha |Du|(X).$$

Non-Local Functionals

Let $p \in [1,\infty)$ and $f : X \times X \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ be a $\mathcal{L}^m \times \mathcal{L}^m \times \mathcal{B} \times \mathcal{B}$-measurable function with the symmetry

$$f(x,y,w,z) = f(y,x,z,w) \quad \text{for all} \quad x,y \in X,\ w,z \in \mathbb{R}.$$

Definition (Non-Local Functional)

Then we call

$$\mathcal{F} : L^p(X) \to \mathbb{R} \cup \{\infty\}, \quad \mathcal{F}(u) = \int_X \int_X f(x,y,u(x),u(y))\,\mathrm{d}x\,\mathrm{d}y$$

the non-local functional on $L^p(X)$ defined by f.

A problem in this formulation is that different functions f may define the same non-local functional $\mathcal{F}$. E.g. $f_1 \equiv 0$ and $f_2(x,y,w,z) = (\frac{1}{2} - y)w^2 + (\frac{1}{2} - x)z^2$ define both the zero functional on $L^p([0,1])$.

- J. Boulanger, P. Elbau, C. Pontow, and O. Scherzer.
  Non local functionals for imaging.
  submitted.

Derivative-Free Formulation of the Sobolev Semi-Norm

Let $q \in [1, \infty)$ and $(\delta_\ell)_{\ell \in \mathbb{N}} \subset L^1(\mathbb{R}^m; [0, \infty))$ be a δ-sequence of radially symmetric, monotonically decreasing functions.
Then we know from [Bourgain, Brézis, and Mironescu, 2002] and [Ponce, 2004] that for measurable functions $u : X \to \mathbb{R}$

$$\mathcal{R}_\ell(u) = \int_X \int_X \frac{|u(x) - u(y)|^q}{|x - y|^q} \delta_\ell(x - y)\, \mathrm{d}x\, \mathrm{d}y$$

is an approximation for the Sobolev-semi-norm in the sense that there exists a constant $K_q \in \mathbb{R}$ such that

$$\begin{aligned} \lim_{\ell \to \infty} K_q \mathcal{R}_\ell(u) &= \int_X |\nabla u(x)|^q \,\mathrm{d}x \qquad &&\text{if} \quad q \in (1, \infty) \quad \text{and} \quad u \in W^{1,q}(X), \\ \lim_{\ell \to \infty} K_1 \mathcal{R}_\ell(u) &= |Du|(X) &&\text{if} \quad q = 1 \quad \text{and} \quad u \in BV(X). \end{aligned}$$

Otherwise, $\lim_{\ell \to \infty} \mathcal{R}_\ell(u) = \infty$.

- J. Bourgain, H. Brézis, and P. Mironescu.
  Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications.
  Journal d'Analyse Mathématique, 87(1):77–101, 2002.
- A.C. Ponce.
  A new approach to sobolev spaces and connections to Γ-convergence.
  Calculus of Variations and Partial Differential Equations, 19(3):229–255, 2004.

Derivative-Free Formulation of Total Variation Denoising

We may therefore approximate for instance the total variational functional

$$\mathcal{E} : L^2(X) \to \mathbb{R} \cup \{\infty\}, \quad \mathcal{E}(u) = \mathcal{S}(u - u_0) + \alpha |Du|(X),$$

where $\mathcal{S}$ denotes a functional measuring the distance to the original image u_0, e.g. $\mathcal{S}(u - u_0) = \|u - u_0\|_2^2$, and $\alpha \in [0, \infty)$ is some regularisation parameter, by the non-local functional $\mathcal{F}_\ell : L^2(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$\mathcal{F}_\ell(u) = \mathcal{S}(u - u_0) + \tilde{\alpha} \int_X \int_X \frac{|u(x) - u(y)|}{|x - y|} \delta_\ell(x - y)\, \mathrm{d}x\, \mathrm{d}y$$

for some large value $\ell \in \mathbb{N}$ and $\tilde{\alpha} = K_1 \alpha$.

Indeed, it is shown in [Aubert, Kornprobst, 2009] and [Pontow, Scherzer, 2009] (for suitable functionals $\mathcal{S}$) that if $u_\ell \in L^2(X)$ is a minimising point of $\mathcal{F}_\ell$, then $(u_\ell)_{\ell \in \mathbb{N}}$ has a subsequence converging in the L^1-norm to the minimising point of $\mathcal{E}$.

- G. Aubert and P. Kornprobst.
  Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful for solving variational problems?
  SIAM Journal on Numerical Analysis, 47:844, 2009.
- C. Pontow and O. Scherzer.
  A derivative-free approach to total variation regularization.
  Arxiv preprint arXiv:0911.1293, 2009.

Non-Local Gradients

A similar approach can be found in [Gilboa, Osher, 2008]. They define a non-local gradient

$$\nabla_w u(x,y) = (u(x) - u(y))\sqrt{w(x,y)},$$

depending on some non-negative weight function w and consider $\nabla_w u(x,\cdot)$ as vector in $L^2(X)$ so that

$$|\nabla_w u(x,\cdot)| = \sqrt{\langle \nabla_w u(x,\cdot), \nabla_w u(x,\cdot)\rangle_{L^2(X)}} = \sqrt{\int_X |\nabla_w u(x,y)|^2 \,\mathrm{d}y}.$$

Then replacing $|\nabla u(x)|$ by $|\nabla_w u(x,\cdot)|$, the total variation functional would for instance become

$$\mathcal{F}(u) = \mathcal{S}(u - u_0) + \alpha \int_X \sqrt{\int_X (u(x) - u(y))^2 w(x,y)\,\mathrm{d}y}\,\mathrm{d}x,$$

which is too general for our definition of a non-local functional.

- Guy Gilboa and Stanley Osher.
  Nonlocal operators with applications to image processing.
  Multiscale Model. Simul., 7(3):1005–1028, 2008.

Definition of Bilateral Filters

Let $g : [0, \infty) \to [0, \infty)$ and $k : [0, \infty) \to [0, \infty)$ be bounded functions.

Definition (Bilateral Filter)

We call a transformation $\mathcal{N}_{k,g}$ of the form

$$\mathcal{N}_{k,g}(u_0)(x) = \frac{1}{C} \int_X k(|x - y|) g(|u_0(x) - u_0(y)|^2) u_0(y) \, \mathrm{d}y, \quad x \in X,$$

with $C = \int_X k(|x - y|) g(|u_0(x) - u_0(y)|^2) \, \mathrm{d}y$ a bilateral filter.

These filters were introduced in [Yaroslavsky, Yaroslavskij, 1985] with

$$k(\Delta x) = \chi_{[0,\varrho]}(\Delta x) \quad \text{and} \quad g(\Delta u^2) = \mathrm{e}^{-\frac{1}{2\sigma_r^2} \Delta u^2}.$$

Another common choice is to use also for k a Gaussian function: $k(\Delta x) = \mathrm{e}^{-\frac{1}{2\sigma_d^2} \Delta x^2}$, see [Tomasi, Manduchi, 1998].

- LP Yaroslavsky and LP Yaroslavskij.
  Digital picture processing. An introduction.
  Springer, 1985.
- C. Tomasi and R. Manduchi.
  Bilateral filtering for gray and color images.
  In *Proceedings of the Sixth International Conference on Computer Vision*, volume 846, 1998.

Non-Local Functionals and Bilateral Filters

We consider for a differentiable function G with bounded derivative the non-local functional

$$\mathcal{R} : L^2(X) \to \mathbb{R}, \quad \mathcal{R}(u) = \int_X \int_X k(|x-y|) G(|u(x)-u(y)|^2) \,\mathrm{d}x \,\mathrm{d}y.$$

Then the directional derivative $\delta\mathcal{R}(u;v)$ in the direction of a function v is

$$\delta\mathcal{R}(u;v) = 4 \int_X \int_X k(|x-y|) G'(|u(x)-u(y)|^2)(u(x)-u(y)) v(x) \,\mathrm{d}x \,\mathrm{d}y.$$

So the condition $\delta\mathcal{R}(u;v) = 0$ for all $v \in L^2(X)$ for u to be a critical point of $\mathcal{R}$ becomes

$$u(x) = \frac{\int_X k(|x-y|) G'(|u(x)-u(y)|^2) u(y) \,\mathrm{d}y}{\int_X k(|x-y|) G'(|u(x)-u(y)|^2) \,\mathrm{d}y}, \quad x \in X.$$

Solving this equation with a fixed-point iteration with initial data u_0, the first step is the bilateral filter $\mathcal{N}_{k,G'}$.

Variational Formulation of Bilateral Filters

So, instead of applying the neighbourhood filter $\mathcal{N}_{k,g}$, we may think of minimising the regularisation functional $\mathcal{F} : L^2(X) \to \mathbb{R} \cup \{\infty\}$

$$\mathcal{F}(u) = \mathcal{S}(u - u_0) + \alpha \int_X \int_X k(|x - y|) G(|u(x) - u(y)|^2) \, \mathrm{d}x \, \mathrm{d}y,$$

where G is a primitive function of g, $\alpha \in (0, \infty)$ is some regularisation parameter, and $\mathcal{S} : L^2(X) \to \mathbb{R} \cup \{\infty\}$ is a functional measuring the distance to the original image u_0, e.g.

$$\mathcal{S}(u - u_0) = \int_X |u(x) - u_0(x)|^2 \, \mathrm{d}x,$$

see [Kindermann, Osher, Jones, 2006].

- S. Kindermann, S. Osher, and P.W. Jones.
Deblurring and denoising of images by nonlocal functionals.
Multiscale Modeling and Simulation, 4(4):1091–1115, 2006.

Definition of Patch-Based Filters

Instead of only comparing the intensity between two different points, we may want to compare the intensities between two patches around the points.

Definition

We call a map $\mathcal{N}_{k,g,h}$ of the form

$$\mathcal{N}_{k,g,h}(u_0)(x) = \frac{1}{C}\int_X k(|x-y|)g(H_{u_0}(x,y))u_0(y)\,\mathrm{d}y,$$

where $H_{u_0}(x,y) = \int_X h(t)(u_0(x+t)-u_0(y+t))^2\,\mathrm{d}t$ measures the difference of the intensities of the patches and $C = \int_X k(|x-y|)g(H_{u_0}(x,y))\,\mathrm{d}y$, a patch-based filter.

The best known example of this method is probably the Non-Local Means Filter where g and h are Gaussian functions, see [Buades, Coll, Morel, 2006].

- A. Buades, B. Coll, and J.M. Morel.
  A review of image denoising algorithms, with a new one.
  Multiscale Modeling and Simulation, 4(2):490–530, 2006.
- J. Boulanger, C. Kervrann, J. Salamero, J.-B. Sibarita, P. Elbau, and P. Bouthemy.
  Patch-based non-local functional for denoising fluorescence microscopy image sequences.
  IEEE Transactions on Medical Imaging, 2010.

Variational Formulation of Patch-Based Filters

Similar to bilateral filters, we consider the non-local functional

$$\mathcal{R} : L^2(X) \to \mathbb{R} \cup \{\infty\}, \quad \mathcal{R}(u) = \int_X \int_X k(|x-y|) G(H_u(x,y)) \, \mathrm{d}x \, \mathrm{d}y.$$

Then a critical point $u \in L^2(X)$ fulfils

$$u(x) = \frac{\int_X \tilde{g}(x,y,u) k(x,y) u(y) \, \mathrm{d}y}{\int_X \tilde{g}(x,y,u) k(x,y) \, \mathrm{d}y}, \quad x \in X,$$

where

$$\tilde{g}(x,y,u) = \int_X h(s) G'(H_u(x+s, y+s)) \, \mathrm{d}s.$$

Seeing $\tilde{g}$ as a smoothed version of G', the functional

$$\mathcal{F}(u) = \mathcal{S}(u-u_0) + \alpha \int_X \int_X k(|x-y|) G(H_u(x,y)) \, \mathrm{d}x \, \mathrm{d}y$$

may be considered as an (approximate) variational formulation of the patch-based filter $\mathcal{N}_{k,G',h}$, see again [Kindermann, Osher, Jones, 2006]

- S. Kindermann, S. Osher, and P.W. Jones.
  Deblurring and denoising of images by nonlocal functionals.
  Multiscale Modeling and Simulation, 4(4):1091–1115, 2006.

Local versus Non-Local Methods

Noisy Image

Gaussian Convolution

Anisotropic Diffusion

Total Variation

Bilateral Filter

Non-Local Means Filter

- Antoni Buades, Bartomeu Coll, and Jean-Michel Morel.
  A non-local algorithm for image denoising.
  In *In CVPR*, pages 60–65, 2005.

Criterion for the Existence of Minimising Points

Let $\mathcal{F} : L^p(X;\mathbb{R}^n) \to \mathbb{R} \cup \{\infty\}$ be a functional, $p \in (1,\infty)$, $n \in \mathbb{N}$.

Definition

The functional $\mathcal{F}$ is called coercive if

$$\lim_{\ell\to\infty} \mathcal{F}(u_\ell) = \infty \quad \text{whenever} \quad \lim_{\ell\to\infty} \|u_\ell\|_p = \infty.$$

It is called weakly sequentially lower semi-continuous if

$$\liminf_{\ell\to\infty} \mathcal{F}(u_\ell) \geq \mathcal{F}(u) \quad \text{whenever} \quad u_\ell \rightharpoonup u.$$

These two properties may be verified to guarantee a minimising point.

Theorem (Existence of Minimising Points)

If $\mathcal{F}$ is coercive and weakly sequentially lower semi-continuous, then $\mathcal{F}$ has a minimising point.

Lower Semi-Continuity of Local Functionals

For local functionals, a precise characterisation of weakly sequentially lower semi-continuous functionals can be found e.g. in [Dacorogna, 1982] or [Fonseca, Leoni, 2007].

Theorem

Let $e : X \times \mathbb{R}^n \to [0, \infty)$ be a $\mathcal{L}^m \times \mathcal{B}^n$-measurable function such that the function $w \mapsto e(x, w)$ is lower semi-continuous, $n \in \mathbb{N}$. Then the functional

$$\mathcal{E} : L^p(X; \mathbb{R}^n) \to \mathbb{R} \cup \{\infty\}, \quad \mathcal{E}(u) = \int_X e(x, u(x))\,\mathrm{d}x$$

is weakly sequentially lower semi-continuous if and only if $w \mapsto e(x, w)$ is for almost all $x \in X$ convex.

- B. Dacorogna.
  Weak Continuity and Weak Lower Semicontinuity for Nonlinear Functional, volume 922 of *Lecture Notes in Mathematics*.
  Springer, 1982.
- I. Fonseca and G. Leoni.
  Modern methods in the calculus of variations: L^p spaces.
  Springer, 2007.

Weak Lower Semi-Continuity of Non-Local Functionals

Let $\mathcal{F}$ be a non-local functional on $L^p(X)$ defined by the function $f : X \times X \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$. Similarly to the local case, we find (under more regularity conditions) an equivalent characterisation for the weak lower semi-continuity of $\mathcal{F}$, see [Elbau, 2010]. We assume that

- $f(\cdot, \cdot, w, z) \in L^\infty(X \times X)$ for all $w, z \in \mathbb{R}$,
- $f(x, y, \cdot, \cdot) \in C^2(\mathbb{R}^2)$, and
- the partial derivatives $\partial_w f(x, \cdot, w, z)$ and $\partial_w^2 f(x, \cdot, w, z)$ are locally integrable.

Theorem

Then $\mathcal{F}$ is weakly sequentially lower semi-continuous if and only if there exists a function $\tilde{f} : X \times X \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ such that

- *$\tilde{f}(x, y, \cdot, \cdot)$ is separately convex for almost all $x, y \in X$ and*
- *the non-local functional $\mathcal{F}$ is defined by $\tilde{f}$.*

P. Elbau.
Sequential lower semi-continuity of non-local functionals.
preprint, 2010.

Sketch of the Proof (Part 1)

Let $\mathcal{F}$ be weakly lower semi-continuous. We pick some $w_1, w_2 \in \mathbb{R}$ and a $\vartheta \in [0, 1]$ and choose characteristic functions χ_{E_ℓ} converging weakly-star in $L^\infty(X)$ to the constant function ϑ.

Then for every $\psi \in L^p(X)$ the functions $u_k \in L^p(X)$ given by

$$u_k(x) = \varphi_k(x)\chi_A(x) + \psi(x)\chi_{A^c}(x), \quad \varphi_k(x) = w_2 + \chi_{E_k}(x)(w_1 - w_2),$$

weakly converge to the function $u \in L^p(X)$ given by

$$u(x) = \bar{w}\chi_A(x) + \psi(x)\chi_{A^c}(x), \quad \bar{w} = \vartheta w_1 + (1 - \vartheta)w_2.$$

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weakly converge to the function $u \in L^p(X)$ given by

$$u(x) = \bar{w}\chi_A(x) + \psi(x)\chi_{A^c}(x), \quad \bar{w} = \vartheta w_1 + (1-\vartheta)w_2.$$

Plugging this into the inequality $\liminf_{k\to\infty} \mathcal{F}(u_k) \geq \mathcal{F}(u)$, we find in the limit $\mathcal{L}^m(A) \to 0$ that the function

$$\Phi_{x,\psi} : \mathbb{R} \to \mathbb{R} \cup \{\infty\}, \quad \Phi_{x,\psi}(w) = \int_X f(x,y,w,\psi(y))\,\mathrm{d}y,$$

is for almost all $x \in X$ and all $\psi \in L^p(X)$ convex.

Sketch of the Proof (Part 2)

Defining the function

$$\tilde{\gamma}(x,y,w) = \inf_{z\in\mathbb{R}} \partial_w^2 f(x,y,w,z),$$

we find from the convexity of $\Phi_{x,z}$ that

$$\int_X \tilde{\gamma}(x,y,w)\,\mathrm{d}y \geq 0.$$

Therefore, there exists a function $\gamma \leq \tilde{\gamma}$ with $\int_X \gamma(x,y,w)\,\mathrm{d}y = 0$.
Setting now

$$\bar{f}(x,y,w,z) = f(x,y,w,z) - \int_0^w \int_0^{\hat{w}} \gamma(x,y,\tilde{w})\,\mathrm{d}\tilde{w}\,\mathrm{d}\hat{w},$$

the function $w \mapsto \bar{f}(x,y,w,z)$ is convex for almost all $x,y \in X$ and all $z \in \mathbb{R}$ and $\mathcal{F}_{\bar{f}} = \mathcal{F}$. Proceeding in the same way for the last component, we get $\tilde{f}$.

The proof that a separately convex integrand defines a weakly lower semi-continuous functional is skipped.

Derivative Free Models

Take the non-local functional

$$\mathcal{F}(u) = \mathcal{S}(u - u_0) + \alpha \int_X \int_X \frac{|u(x) - u(y)|^q}{|x - y|^q} \varphi_\ell(x - y)\, \mathrm{d}x\, \mathrm{d}y,$$

where we assume that

$$\mathcal{S}(u - u_0) = \int_X g(x, u(x) - u_0(x))\, \mathrm{d}x.$$

If for all values $x \in X$, there exists a $y \in X$ such that $\varphi_\ell(x - y) = 0$, the second term cannot compensate any non-convexity of the first one, thus forcing us to choose the function $w \mapsto g(x, w)$ convex.
Let us compare this result with the corresponding local functional

$$\mathcal{E}(u) = \int_X g(x, u(x) - u_0(x))\, \mathrm{d}x + \alpha \int_X |\nabla u(x)|^q\, \mathrm{d}x$$

defined on $W^{1,q}(X)$.

Comparison to the Local Functional

To this end, we recall the characterisation of lower semi-continuous functionals on Sobolev spaces going back to [Berkowitz, 1974].

Theorem

Let $e : X \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty)$ *be a Carathéodory function such that the function* $A \mapsto f(x, w, A)$ *is convex. Then the functional*

$$\mathcal{J} : W^{1,q}(X) \to \mathbb{R} \cup \{\infty\}, \quad \mathcal{J}(u) = \int_X e(x, u(x), \nabla u(x))\,\mathrm{d}x$$

is weakly sequentially lower semi-continuous.

So, the functional $\mathcal{E} : W^{1,q}(X) \to \mathbb{R} \cup \{\infty\}$ is for all Carathéodory functions g weakly lower semi-continuous whereas the derivative-free variant $\mathcal{F} : L^p(X) \to \mathbb{R} \cup \{\infty\}$ is only lower semi-continuous if $w \mapsto g(x, w)$ is convex for almost all $x \in X$.

- L.D. Berkowitz.
  Lower semi-continuity of integral functionals.
  Trans. Amer. Math. Soc., 192:51–57, 1974.

Variational Formulation of the Bilateral Filter

We have seen that a non-local functional corresponding to the Gaussian bilateral filter

$$\mathcal{N}(u_0)(x) = \frac{\int_X k(|x-y|)\mathrm{e}^{-(u_0(x)-u_0(y))^2} u_0(y)\,\mathrm{d}y}{\int_X k(|x-y|)\mathrm{e}^{-(u_0(x)-u_0(y))^2}\,\mathrm{d}y}$$

is of the form

$$\mathcal{F}(u) = \mathcal{S}(u-u_0) + \alpha \int_X \int_X k(|x-y|)(1-\mathrm{e}^{-(u(x)-u(y))^2})\,\mathrm{d}x\,\mathrm{d}y.$$

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is of the form

$$\mathcal{F}(u) = \mathcal{S}(u-u_0) + \alpha \int_X \int_X k(|x-y|)(1-\mathrm{e}^{-(u(x)-u(y))^2})\,\mathrm{d}x\,\mathrm{d}y.$$

Since the function $w \mapsto 1 - \mathrm{e}^{-(w-z)^2}$ is not convex, we should compensate the lack of convexity with the functional $\mathcal{S}$. We may choose

$$\mathcal{S}(u-u_0) = 2\mathcal{L}^m(X) \int_X (u(x)-u_0(x))^2\,\mathrm{d}x.$$

Then $\mathcal{F}$ is the non-local functional defined by the function

$$f(x,y,w,z) = (w-u_0(x))^2 + (z-u_0(y))^2 + \alpha k(|x-y|)(1-\mathrm{e}^{-(w-z)^2}),$$

which is for $\alpha\|k\|_\infty \leq 1$ seperately convex in w and z, but still $\mathcal{F}$ is not convex unless we choose $\alpha\|k\|_\infty \leq \frac{1}{2}$.

Non-Local Functionals on Sobolev Spaces

In the same way as for functionals on Lebesgue-spaces, it is possible to show that separately convex integrands lead to weakly lower semi-continuous functionals, see [Pedregal, 1997].

Theorem

Let $p \in (1,\infty)$ and $f : X \times X \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to [0,\infty)$ be a Carathéodory function such that the map $(A,B) \mapsto f(x,y,w,z,A,B)$ is separately convex. Then the functional $\mathcal{F} : W^{1,p}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$\mathcal{F}(u) = \int_X \int_X f(x,y,u(x),u(y),\nabla u(x),\nabla u(y))\,\mathrm{d}x\,\mathrm{d}y$$

is weakly sequentially lower semi-continuous.

- P. Pedregal.
  Nonlocal variational principles.
  Nonlinear Analysis, 29(12):1379–1392, 1997.

Non-Local Functionals for Vector Valued Functions

For non-local functionals on $L^p(X;\mathbb{R}^n)$, $n \in \mathbb{N}$, we still have that the separate convexity of the integrand implies the weak lower semi-continuity.

Theorem

Let $f : X \times X \times \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ be a Carathéodory function such that the map $(w,z) \mapsto f(x,y,w,z)$ is separately convex. Then the functional

$$\mathcal{F} : L^p(X;\mathbb{R}^n) \to \mathbb{R} \cup \{\infty\}, \quad \mathcal{F}(u) = \int_X \int_X f(x,y,u(x),u(y))\,\mathrm{d}x\,\mathrm{d}y$$

is weakly sequentially lower semi-continuous.

But in contrast to the local case where this condition is also necessary for the lower semi-continuity, this does not seem to be the case for non-local functionals (even up to equivalence).

Thank you for your attention!