Constructing Test Instances for Sparse Recovery Algorithms

Christian Kruschel and Dirk Lorenz, May 20, 2012

SPEAR -
Sparse Exact and Approximate Recovery

TU Braunschweig
Analysis and Algebra

TU Darmstadt
Discrete Optimization

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Project period: 2011-2014
Homepage: http://www.math.nat.tu-bs.de/mo/spear

Minimization Problems for Sparse Recovery

We consider $A \in \mathbb{R}^{m \times n}, m \leqslant n, b \in \mathbb{R}^m$ and

$$\min_y \|y\|_1 \text{ s.t. } Ay = b, \tag{BP}$$

further for $\lambda \gneq 0$ its denoising variant

$$\min_y \frac{1}{2}\|Ay - b\|_2^2 + \lambda\|y\|_1, \tag{QP$_\lambda$}$$

with

$$\|y\|_p^p := \sum_{i=1}^{n} |y_i|^p, p \in \mathbb{N}\backslash\{0\}.$$

Norm Description

So Many Solvers to Compare...

For (BP):

- SPGL1
- NESTA
- YALL1
- ISAL1
- ...

For (QP_λ):

- ISTA
- FISTA
- SparSA
- FPC
- ...

How to compare ?

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How to compare (e.g. in Matlab) ?

L1TestPack

generates test instances for (QP_λ).

www.tu-braunschweig.de/iaa/personal/lorenz/l1testpack

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- matrix A can be gaussian, bernoulli, partial DCT, heaviside, random orthonormal rows
- solution x^* can be chosen gaussian, bernoulli, different dynamic ranges.
- number of non-zero entries in x^* eligible.
- right side b will be calculated (we'll see later...)

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How to construct test instances?

Sufficient and Necessary Condition for (BP)

Theorem

Let $A = [a_1, \dots, a_n] \in \mathbb{R}^{m \times n}$, $x^* \in \mathbb{R}^n$ and $I = supp(x^*)$. x^* solves

$$\min \|y\|_1 \ s.t.\ Ay = Ax^*$$

uniquely if and only if A_I is injective and there exists $w \in \mathbb{R}^m$

$$\begin{aligned} a_i^T w &= sign(x_i^*), && \text{for } x_i^* \neq 0 \text{ resp. } i \in I, \\ |a_j^T w| &\lneq 1, && \text{for } x_j^* = 0 \text{ resp. } j \in I^C. \end{aligned}$$

Notation

For a given $x^* \in \mathbb{R}^n$ we will denote $I \subset \{1, \dots, n\}$ as the support of x^*, hence $I := supp(x^*) = \{i : x_i^* \neq 0\}$. I^C is the complement of I.

Optimality Conditions for (QP_λ)

Lemma (Fuchs, 2004)

The vector x^* is a unique solution of (QP_λ) *if*

$$a_i^T(b - Ax^*) = \lambda \, \mathrm{sign}(x_i^*), \text{ for } x_i^* \neq 0,$$
$$|a_j^T(b - Ax^*)| \lneq \lambda, \text{ for } x_j^* = 0, \text{ and}$$
$$A_I \text{ injective}$$

holds.

Optimality Conditions for (QP_λ)

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The vector x^* is a solution of (QP_λ) if and only if

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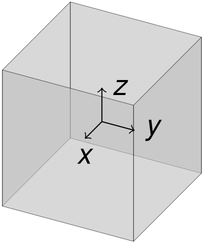
|a_j^T(b - Ax^*)| &\leqslant \lambda, \ \textit{for } x_j^* = 0
\end{aligned}
$$

holds.

$$
\begin{aligned}
\mathrm{Sign}(x^*)_i &= \mathrm{sign}(x_i^*) \text{ if } x_i^* \neq 0 \\
\mathrm{Sign}(x^*)_j &\in [-1, 1] \text{ if } x_j^* = 0
\end{aligned}
$$



Optimality Conditions for (QP_λ)

Lemma (Fuchs, 2004)

The vector x^* is a solution of (QP_λ) if and only if

$$A^T(b - Ax^*) \in \lambda Sign(x^*)$$

holds.

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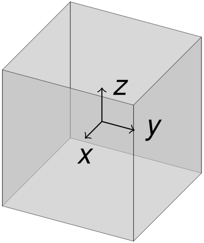

Optimality Conditions for (QP_λ)

Lemma

The vector x^* is a solution of (QP_λ) if and only if

$$A^T(b - Ax^*) \in \lambda \mathit{Sign}(x^*)$$

holds.

Idea

For a given A and $\lambda \gneq 0$, choose x^* and construct b.

Constructing instances for (QP_λ)

Theorem (Lorenz, 2011)

Let $A \in \mathbb{R}^{m \times n}, \lambda \gneq 0, x^* \in \mathbb{R}^n$ and $z \in rg(A^T) \cap Sign(x^*)$.
Then for any w such that $A^T w = z$ and $b = \lambda w + Ax^*$, it holds that x^* solves

$$\min_y \frac{1}{2}\|Ay - b\|_2^2 + \lambda \|y\|_1.$$

Proof.

$$A^T(b - Ax^*) = A^T(\lambda w + Ax^* - Ax^*) = \lambda A^T w \in \lambda \mathrm{Sign}(x^*)$$

$\square$

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Definition (Dual Certificate)

The element $w \in \mathbb{R}^m$ satisfying

$$A^T w \in \mathrm{Sign}(x^*)$$

is called **dual certificate**.

What to construct?

Parameters to solve

$$\min_y \frac{1}{2}\|Ay - b\|_2^2 + \lambda\|y\|_1.$$

uniquely:

- matrix $A \in \mathbb{R}^{m\times n}$,
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- Given A, I^+ and I^-
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Notation

Since I^+ and I^- are prescribed before x^* is chosen,

$$Sign(x^*)$$

is a synonym for I^+ and I^-.

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Constructing instances for (QP_λ)

Idea

For a given $A \in \mathbb{R}^{m\times n}$ and $\lambda \gneq 0$ choose $x^* \in \mathbb{R}^n$ and construct $b \in \mathbb{R}^m$.

- Specify $A \in \mathbb{R}^{m\times n}$, I and restricted sign-vector $z_I, |z_i| = 1, i \in I$.
- Construct $z \in \mathbb{R}^n, |z_j| \lneq 1, j \in I^C$, and solve $A^T w = z$.
- Choose $\lambda \gneq 0$ and x^* according to z and set

$$b = \lambda w + Ax^*.$$

- Vector x^* solves (BP) uniquely, $x^* = \arg\min_y \|y\|_1$ s.t. $Ay = Ax^*$.
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Constructing Dual Certificate

- Method 1:
  Find $z \in \mathrm{rg}(A^T) \cap \mathrm{Sign}(x^*)$, solve $A^T w = z$.
  - by Alternating Projection,
  - $z_j \approx \pm 1$, for some $j \in I^C$.
- Method 2:
  $\min_z \|z_{I^C}\|_2^2$ s.t. $z \in \mathrm{rg}(A^T) \cap \mathrm{Sign}(x^*)$.

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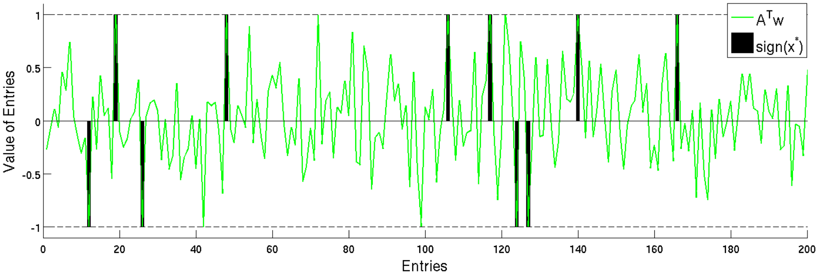

Test: matrix gaussian, size 100×200, solution gaussian, sparsity 10.

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  - Same z as in Method 1. (see Deutsch (2001))

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  - Same z as in Method 1. (see Deutsch (2001))
- Method 2.1:
  $\min_z \|z_{I^C}\|_2^2$ s.t. $z \in \mathrm{rg}(A^T) \cap \mathrm{Sign}(x^*)$, $\|z_{I^C}\|_\infty \leqslant \gamma$.
  - Given $\gamma \in (0, 1)$.

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- Method 2:
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  - Similar to Method 1.
- Method 2.1:
  $\min_z \|z_{I^C}\|_2^2$ s.t. $z \in \mathrm{rg}(A^T) \cap \mathrm{Sign}(x^*)$, $\|z_{I^C}\|_\infty \leqslant \gamma = 0.9$.
  - Might be infeasible, but $\|z_{I^C}\|_\infty \lneq 1$, e.g.

$$A = \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 0.45 & -1 \end{pmatrix}, I^+ = \{1\}, I^- = \{3\}.$$

Decreasing γ

Constructing Dual Certificate

- Method 1:
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- Method 3:
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Constructing Dual Certificate

- Method 3:
  $\min_z \|z_{I^C}\|_\infty$ s.t. $z \in \operatorname{rg}(A^T) \cap \operatorname{Sign}(x^*)$
  - is equivalent to

$$\min_w \|A_{I^C}^T w\|_\infty \text{ s.t. } A_I^T w = \operatorname{sign}(x^*)_I \text{ and}$$

$$\begin{bmatrix} A_{I^C}^T \\ -A_{I^C}^T \end{bmatrix} w \leqslant \begin{pmatrix} \mathbf{1}_{I^C} \\ \mathbf{1}_{I^C} \end{pmatrix},$$

  - is realizable as a Linear Program.

Constructing Dual Certificate - Test

Tested Method: $\min_w \|A_{I^C}^T w\|_\infty$ s.t. $A^T w \in \mathrm{Sign}(x^*)$

A and x^* chosen by L1TestPack

Used Parameter:

- Gaussian distributed matrix with varying size,
- Bernoulli matrix with varying size, entries $= \pm 1$ randomly,
- Partial DCT matrix, generated through uniform sampling of m rows from the full Discrete Cosine Transform matrix.

Measured Values:

- Error on Support, $\|A_I^T w \pm 1\|_2$,
- Optimized value, $\|A_{I^C}^T w\|_\infty$,
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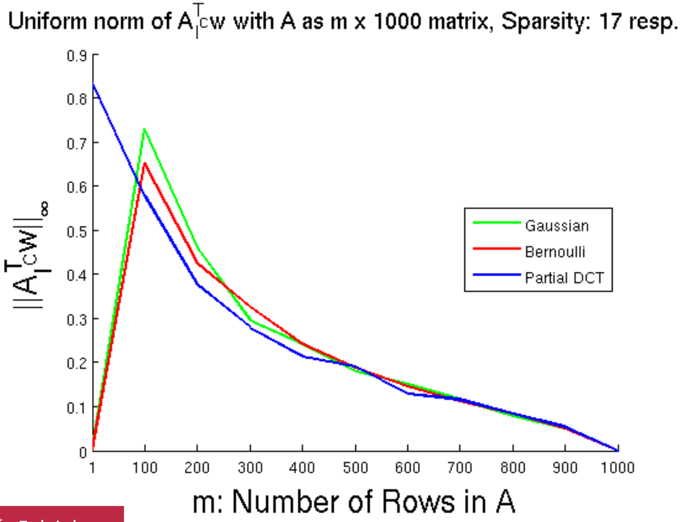

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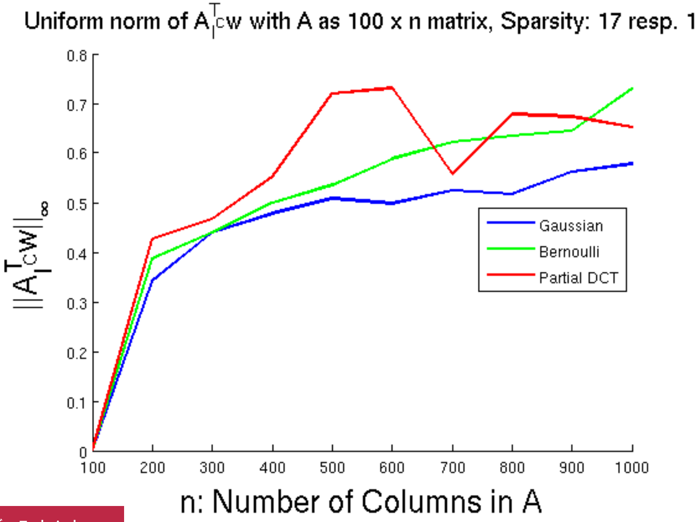

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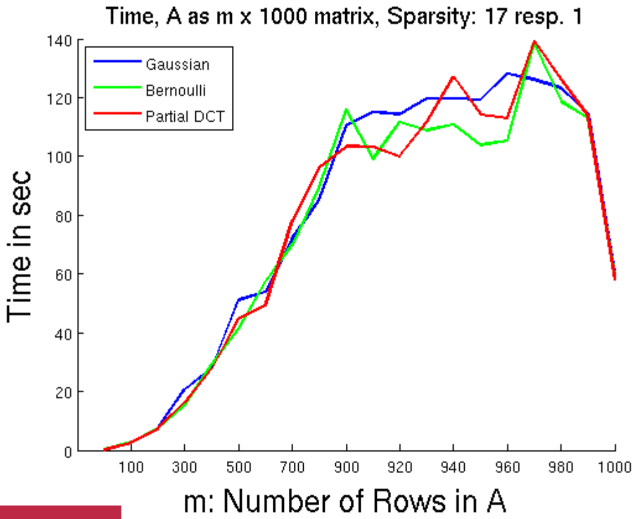

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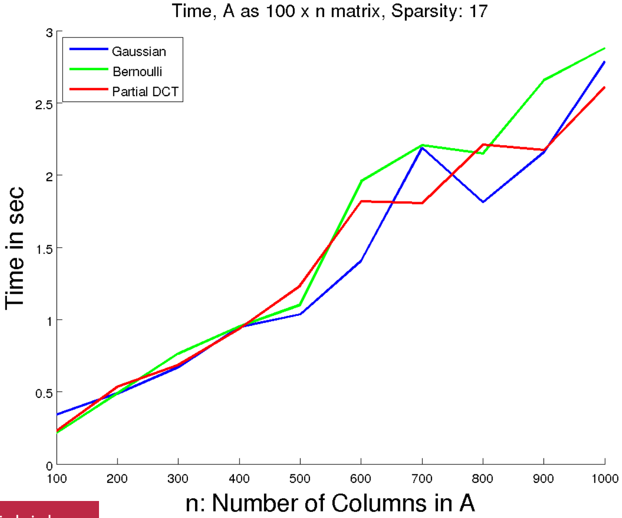

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What to construct?

Construction 2:

- Given A
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Additional Goal:

Find index set $I^+ \cup I^-$ with largest possible cardinality.

What to construct?

Construction 2:

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- Find w, I^+ and I^-.
- Choose x^* and λ and construct $b = \lambda w + Ax^*$.

Additional Goal:

Find index set $I^+ \cup I^-$ with largest possible cardinality.
$\rightarrow$ any other cardinality $\leqslant \operatorname{rank}(A)$ chooseable

Recoverable Indices

Definition

An index set $I \subset \{1, ..., n\}$ is called **recoverable** for $A \in \mathbb{R}^{m \times n}$, if there exists $x^* \in \mathbb{R}^n$, $I = \operatorname{supp}(x^*)$, solving uniquely

$$\min \|y\|_1 \text{ s.t. } Ay = Ax^*.$$

A recoverable index set I is called **maximal**, if there exists no recoverable index set J satisfying $|I| \lneq |J|$.

The index set I is recoverable if there exists $w \in \mathbb{R}^m$ satisfying

$$|a_i^T w| = 1, \text{ for } i \in I,$$

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$$|a_i^T w| = 1, \text{ for } i \in I,$$

$$|a_j^T w| \lneq 1, \text{ for } j \in I^C, \quad \text{Recov. index set } I \text{ depends on } A \text{ and } w.$$

$$A_I \text{ is injective.}$$

Recoverable Indices

Definition

An index set $I \subset \{1, ..., n\}$ is called **recoverable** for $A \in \mathbb{R}^{m \times n}$, if there exists $x^* \in \mathbb{R}^n$, $I = \operatorname{supp}(x^*)$, solving uniquely

$$\min \|y\|_1 \text{ s.t. } Ay = Ax^*.$$

A recoverable index set I is called **maximal**, if there exists no recoverable index set J satisfying $|I| \lneq |J|$.

The index set I is recoverable if there exists $w \in \mathbb{R}^m$ satisfying

$$|a_i^T w| = 1, \text{ for } i \in I,$$

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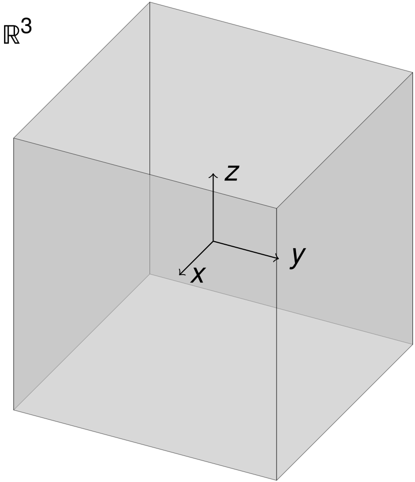
Recov. index set I depends on A and w.

$$A_I \text{ is injective.}$$

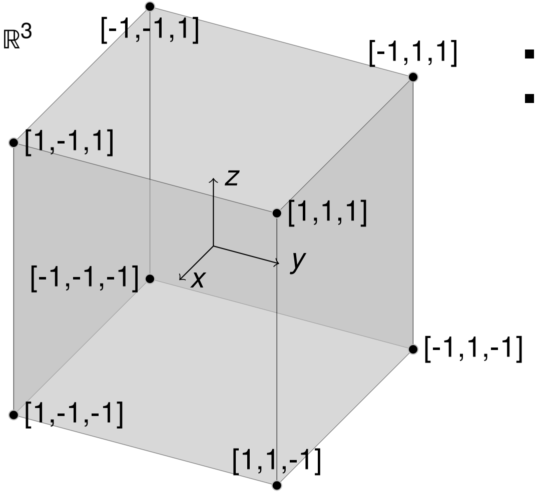
How to compute a maximal index set?

Find Maximal Recoverable Index Set



- Unit Cube in $\mathbb{R}^3$
- Cube = $\{(x, y, z) : |x| \leqslant 1, |y| \leqslant 1, |z| \leqslant 1\}$

Find Maximal Recoverable Index Set



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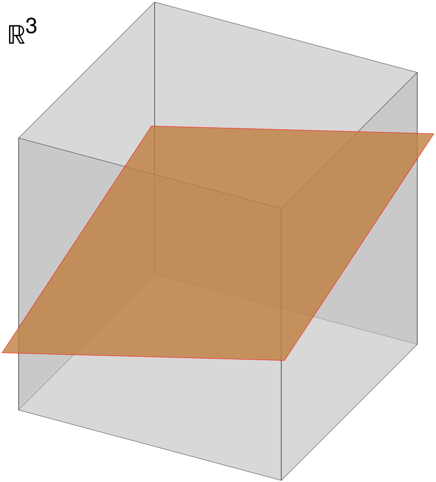
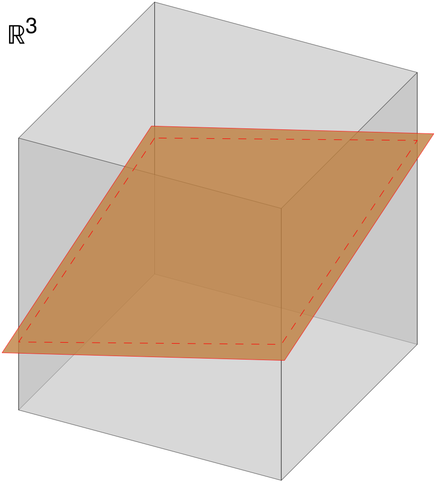


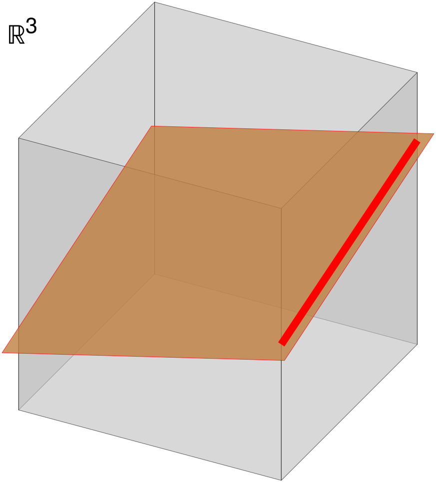
- Range of A^T
- $\mathrm{rg}(A^T) = \{z : \exists w : A^T w = z\}$

Find Maximal Recoverable Index Set



- Intersection of Cube and $\mathrm{rg}(A^T)$
- $\exists z \in \mathrm{Cube} \cap \mathrm{rg} A^T$

Find Maximal Recoverable Index Set



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- Recoverable Index Sets: $I = I^+ \cup I^-$
  - $I^+ = \{2\}$
  - $I^+ = \{2\}, I^- = \{1\}$
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  - etcetera

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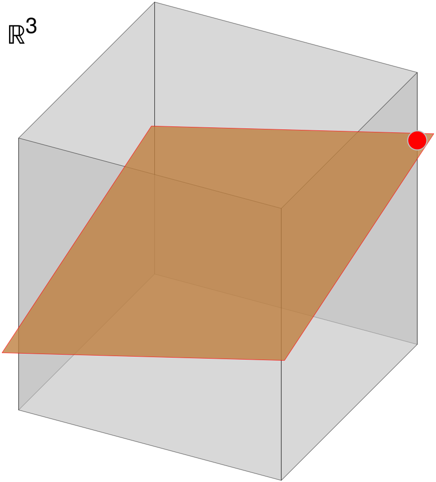

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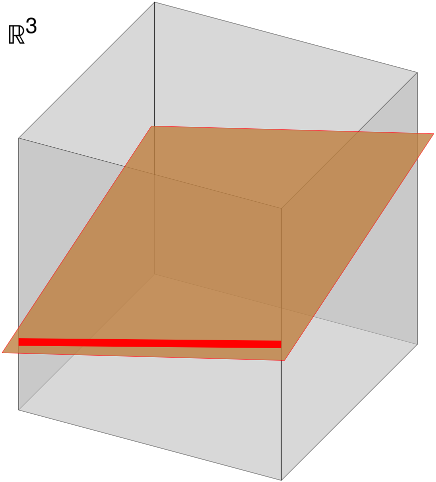

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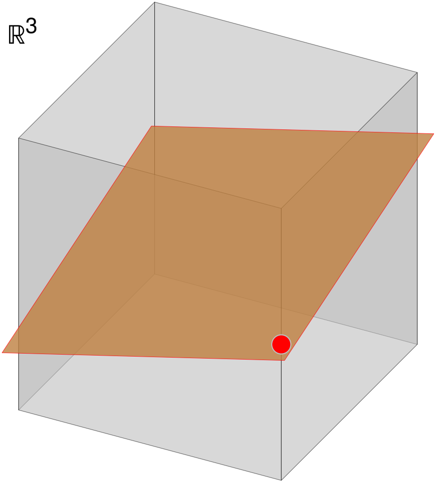

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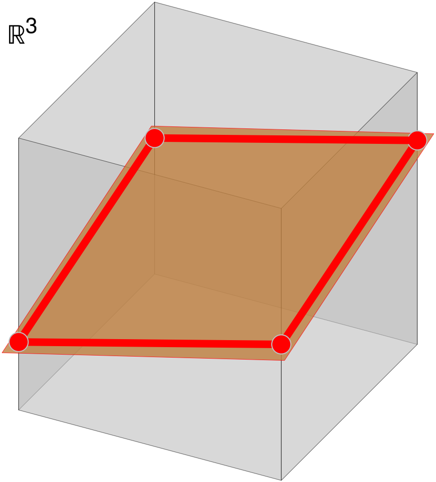

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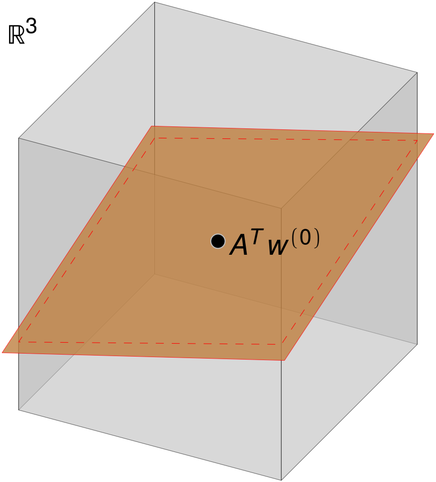
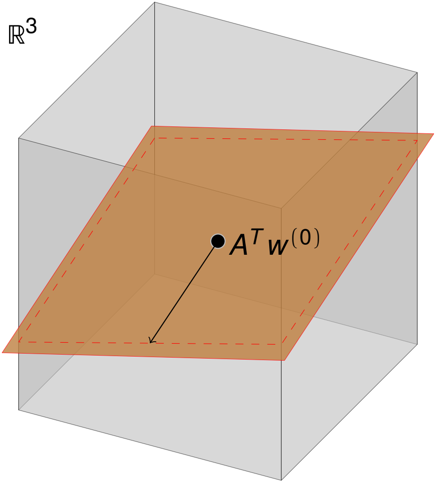


Find Maximal Recoverable Index Set:

- `Start at` $w^{(0)} = (0, 0)$.

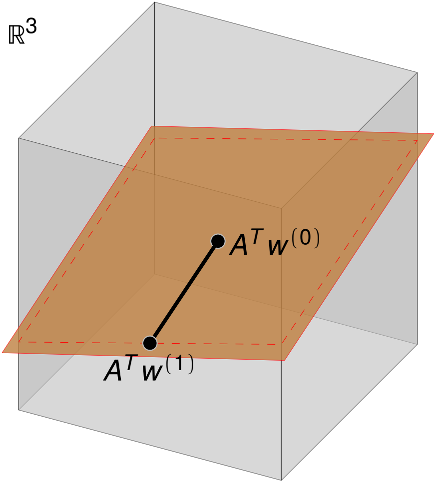
Find Maximal Recoverable Index Set



Find Maximal Recoverable Index Set:

- Start at $w^{(0)} = (0, 0)$.
- Find $j \in \arg\max_i \|a_i\|_2$.

Find Maximal Recoverable Index Set



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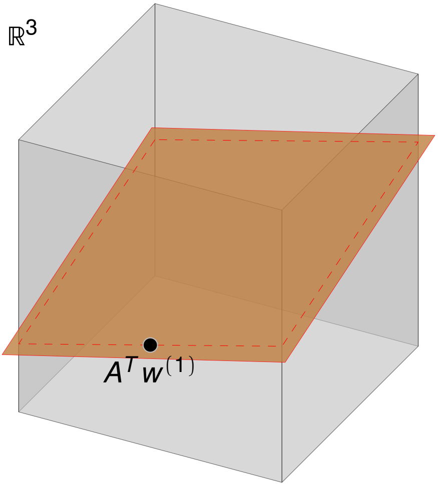

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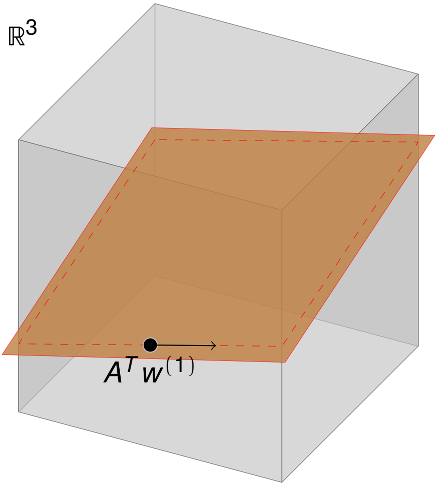

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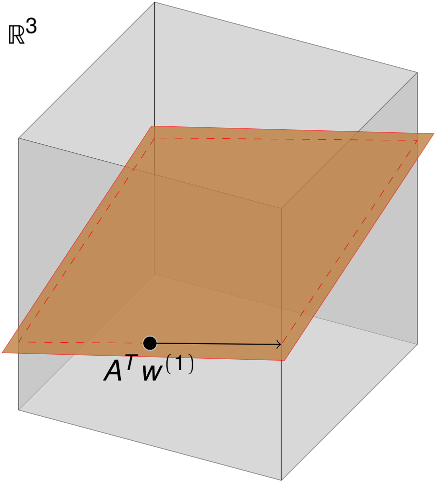

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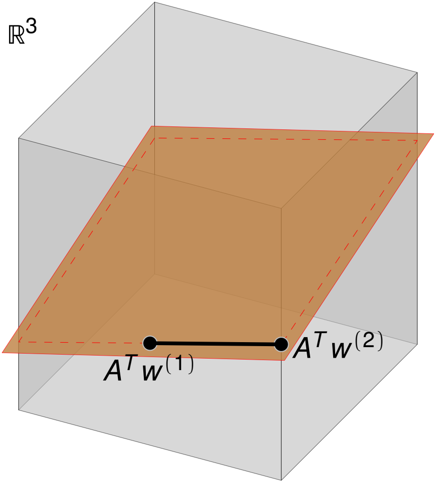
Find Maximal Recoverable Index Set



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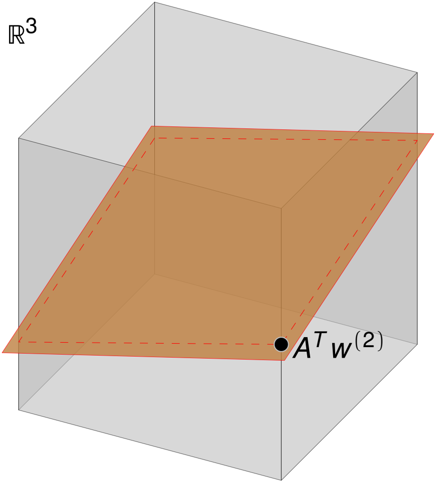

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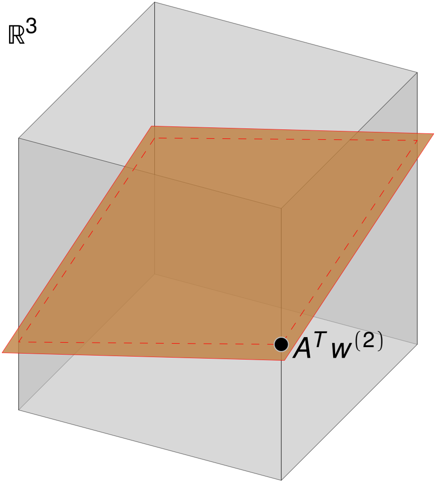

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- Set $w^{(2)} := w^{(1)} + \alpha h_1$ and $I_2 := \{i : |w_i^{(2)}| = 1\}$.
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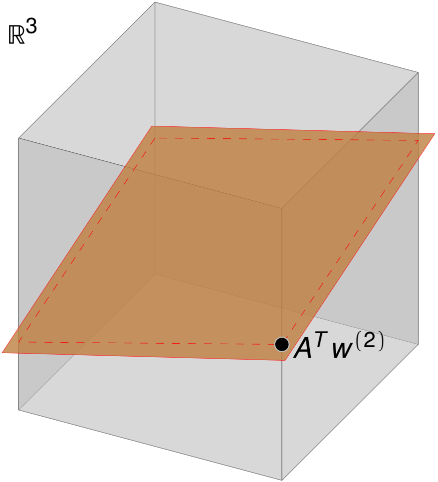
  $\rightarrow$ End Algorithm

Find Maximal Recoverable Index Set



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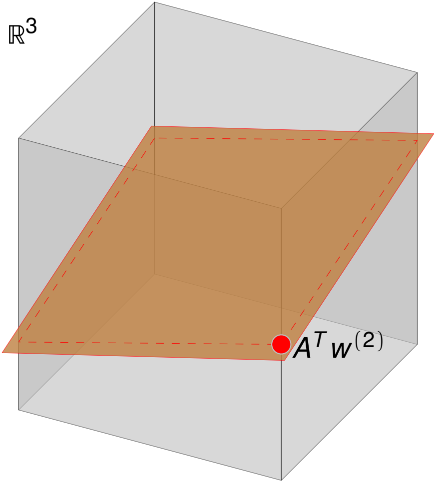

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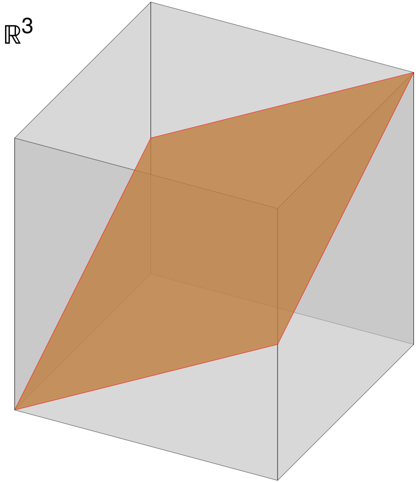
Find Maximal Recoverable Index Set

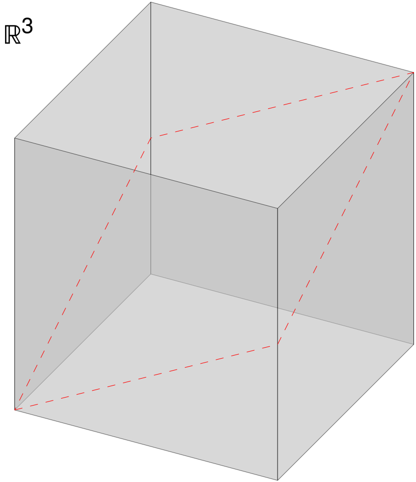


Special Cases:

$$A = \begin{pmatrix} -3 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

-

Find Maximal Recoverable Index Set



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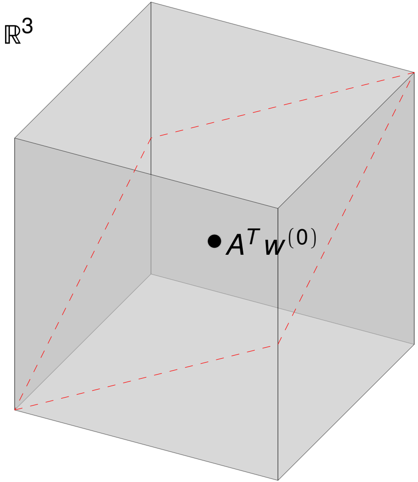

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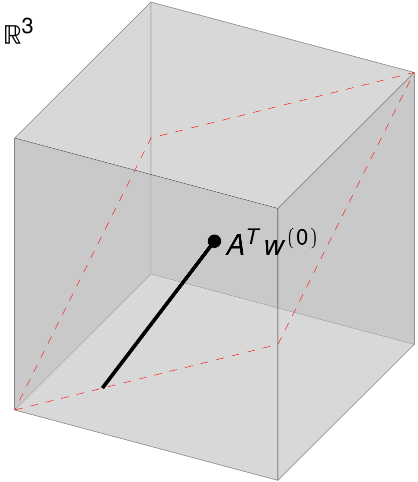

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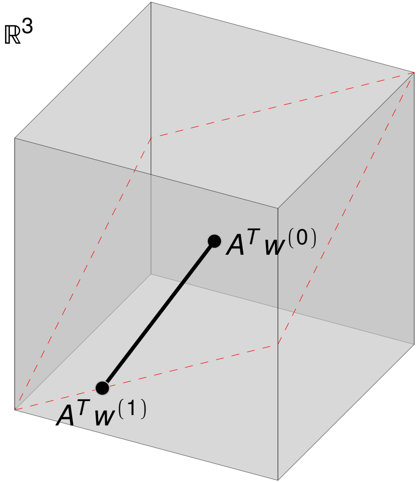

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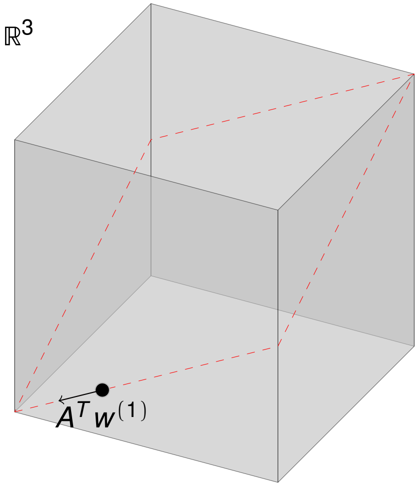

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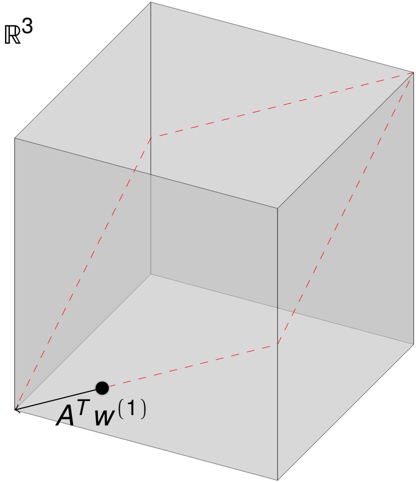

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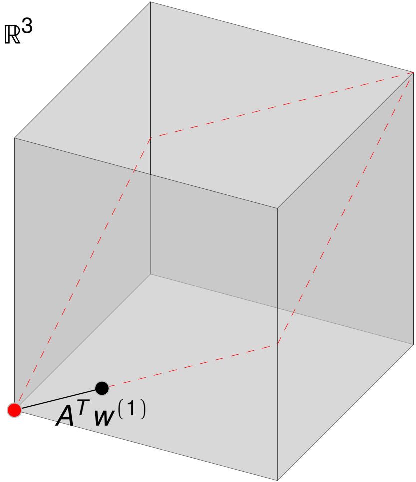

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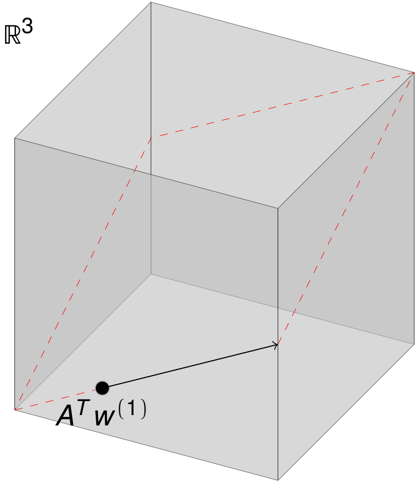
- $|I_2| \not\geq \operatorname{rank}(A_{I_2})$

Find Maximal Recoverable Index Set



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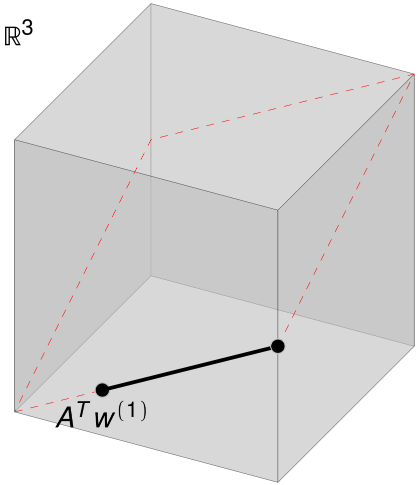

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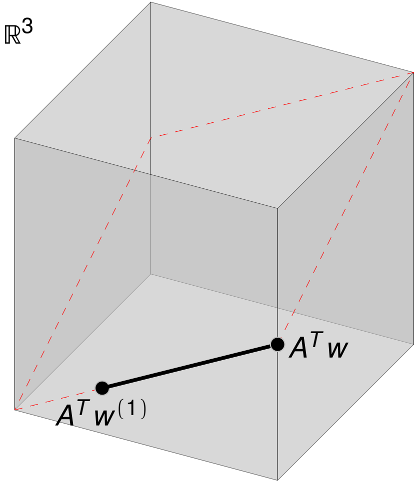
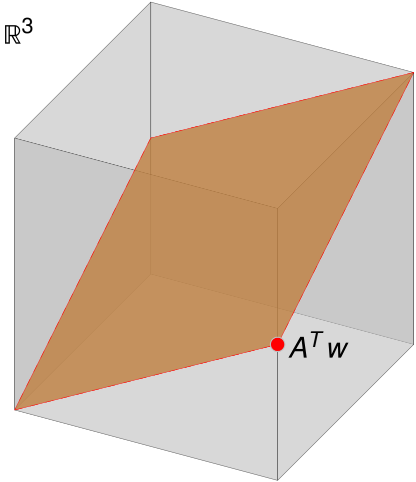


Special Cases:

$$A = \begin{pmatrix} -3 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

- Set $w := w^{(1)} + \alpha h$ and $I := I_3$.

Find Maximal Recoverable Index Set



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Special Case 2

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Find Maximal Recoverable Index Set - Test

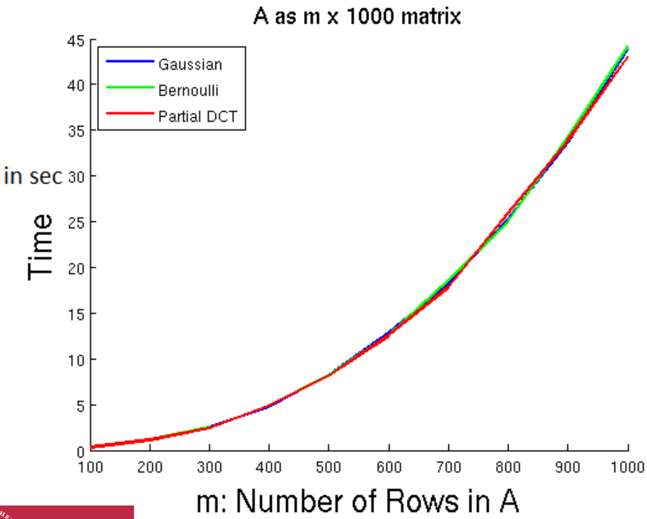
Used Parameter:

- Gaussian distributed matrix with varying size,
- Bernoulli matrix with varying size, entries $= \pm 1$ randomly,
- Partial DCT matrix, generated through uniform sampling of m rows from the full Discrete Cosine Transform matrix.

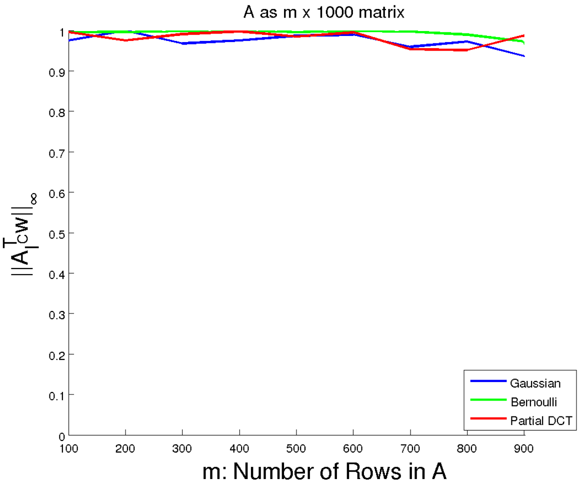
Measured Values:

- Time to get an index set
- Value $\|A_{I^C}^T w\|_\infty$

Find Maximal Recoverable Index Set - Test



Find Maximal Recoverable Index Set - Test



m	$\|A_{I^C}^T w\|_\infty$
100	0.994
200	0.997
300	0.987
400	0.993
500	0.992
600	0.984
700	0.99993
800	0.935
900	0.945

Bernoulli matrix

Conclusion

- One can find test instances for Basis Pursuit / Basis Pursuit Denoising with prescribed sign
- Dual certificate detectable by linear programming or alternating projections
- Found maximal recoverable index set

Future Work

Construction 3:

- Given w, I^+, and I^-
- Find A.
- Choose x^* and λ and construct $b = \lambda w + Ax^*$.

Dual certificate $w \sim \mathcal{N}(0,1)$ could be noise to the right side Ax^*.

Thank You For Your Attention

References:

- F. Deutsch: Best Approximation in Inner Product Spaces,
  CMS Books in Mathematics, 2001 (book)
- J.-J. Fuchs: On sparse representations in arbitrary bases,
  IEEE Transa. o. Info. Theory, 2004 (paper)
- M. Grasmair, M. Haltmeier, O. Scherzer: Necessary and Sufficient Conditions for Linear Convergence of ℓ^1-Regularization,
  Comm. Pure Appl. Math., 2011 (paper)
- A. Jouditski, A. Nemirovski: On Verifiable Sufficient Conditions for Sparse Signal Recovery via ℓ^1 Minimization,
  Mathematical Programming Ser. B, 2011 (paper)
- D. Lorenz: Constructing test imstances for Basis Pursuit Denoising,
  arXiv.org, 2011 (paper)

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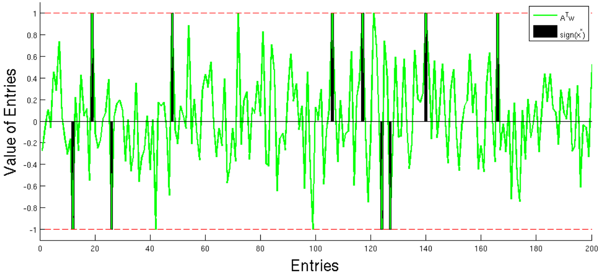
- Facebook, Google+

Constructing Dual Certificate

- Method 2.1:
  $\min_z \|P_{I^c} z\|_2^2$ s.t. $z \in \mathrm{rg}(A^T) \cap \mathrm{Sign}(x^*)$, $\|P_{I^c} z\|_\infty \leqslant \gamma$.
  - Given $\gamma \in [0,1]$.

$$\gamma = 1.0$$



Test: matrix gaussian, size 100×200, sparsity 10.

Back

Constructing Dual Certificate

- Method 2.1:
  $\min_z \|P_{I^c} z\|_2^2$ s.t. $z \in \mathrm{rg}(A^T) \cap \mathrm{Sign}(x^*)$, $\|P_{I^c} z\|_\infty \leqslant \gamma$.
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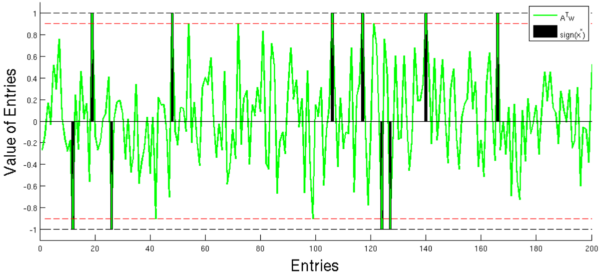
$$\gamma = 0.9$$



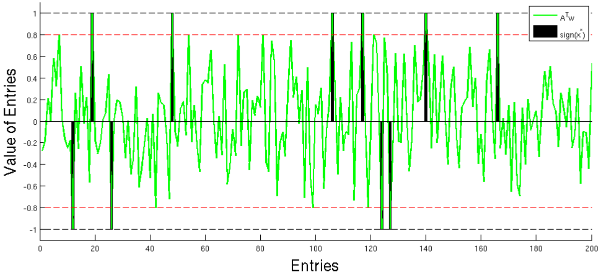
Test: matrix gaussian, size 100×200, sparsity 10.

Back

Constructing Dual Certificate

- Method 2.1:
  $\min_z \|P_{I^c} z\|_2^2$ s.t. $z \in \mathrm{rg}(A^T) \cap \mathrm{Sign}(x^*)$, $\|P_{I^c} z\|_\infty \leqslant \gamma$.
  - Given $\gamma \in [0,1]$.

$$\gamma = 0.8$$



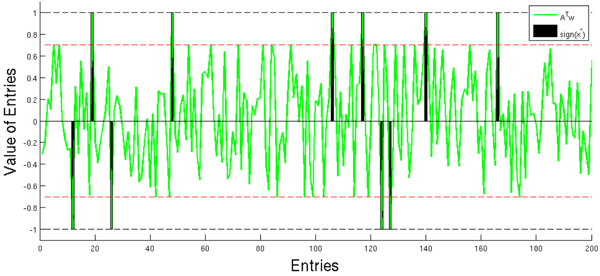
Test: matrix gaussian, size 100×200, sparsity 10.

Back

Constructing Dual Certificate

- Method 2.1:
  $\min_z \|P_{I^c} z\|_2^2$ s.t. $z \in \mathrm{rg}(A^T) \cap \mathrm{Sign}(x^*)$, $\|P_{I^c} z\|_\infty \leqslant \gamma$.
  - Given $\gamma \in [0,1]$.

$$\gamma = 0.7$$



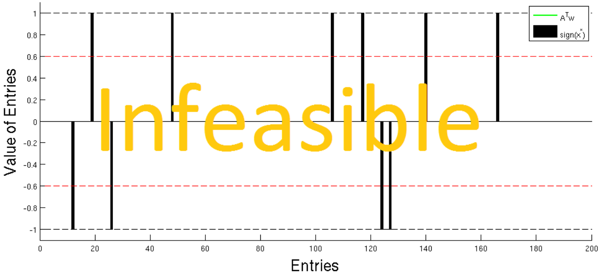
Test: matrix gaussian, size 100×200, sparsity 10.

Back

Constructing Dual Certificate

- Method 2.1:
  $\min_z \|P_{I^c} z\|_2^2$ s.t. $z \in \mathrm{rg}(A^T) \cap \mathrm{Sign}(x^*)$, $\|P_{I^c} z\|_\infty \leqslant \gamma$.
  - Given $\gamma \in [0,1]$.

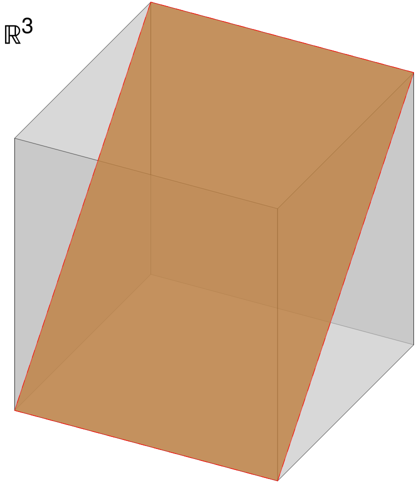
$$\gamma = 0.6$$



Test: matrix gaussian, size 100×200, sparsity 10.

Back

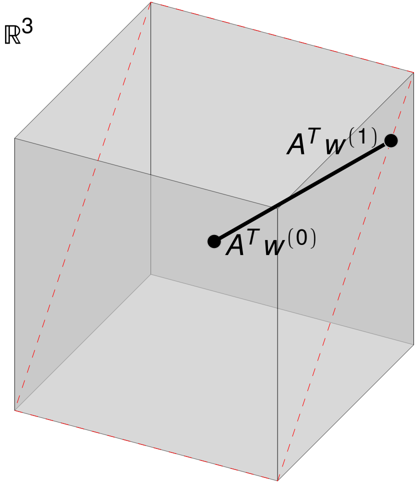
Find Maximal Recoverable Index Set



Special Cases:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & -2 \end{pmatrix}$$

Find Maximal Recoverable Index Set

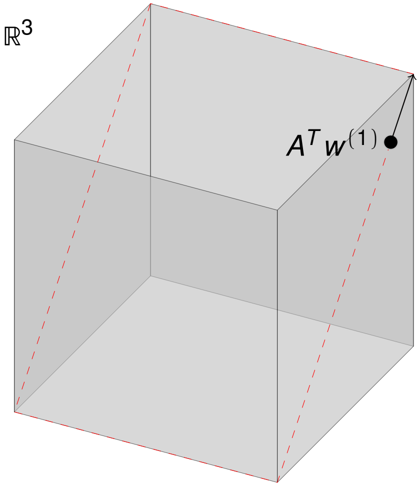


Special Cases:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & -2 \end{pmatrix}$$

- $w^{(1)} = (0, -\frac{1}{3})^T$, $I_1 = \{2\}$

Find Maximal Recoverable Index Set

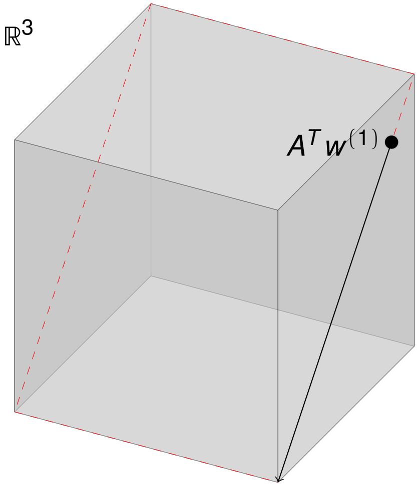


Special Cases:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & -2 \end{pmatrix}$$

- Next possible support $I_2 = [-1, 1, 1]$ leads to A_{I_2} not injective

Find Maximal Recoverable Index Set

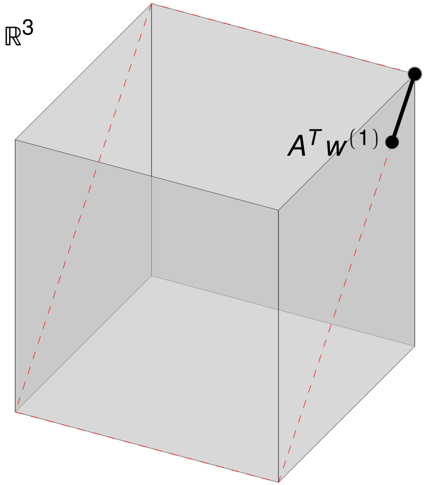


Special Cases:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & -2 \end{pmatrix}$$

- Next possible support $I_3 = [1, 1, -1]$ leads to A_{I_3} not injective

Find Maximal Recoverable Index Set

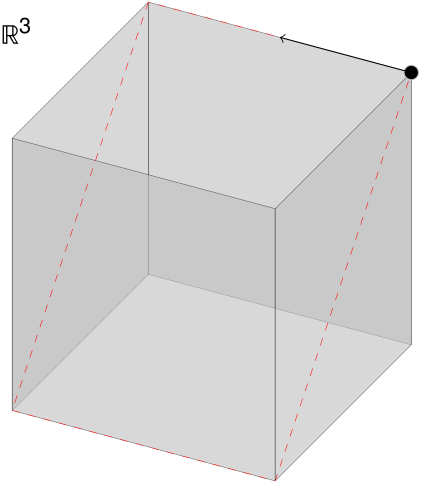


Special Cases:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & -2 \end{pmatrix}$$

- Choose one of these supports

Find Maximal Recoverable Index Set



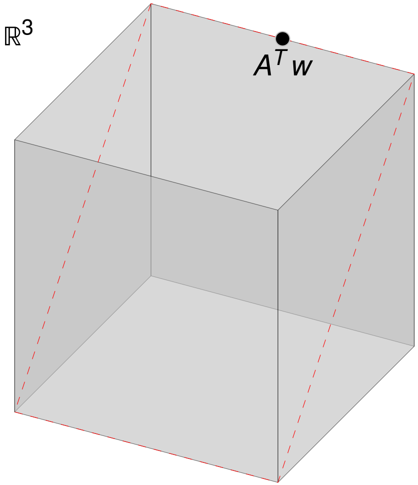
Special Cases:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & -2 \end{pmatrix}$$

- Support $I_4 = [-1, 0, 1]$ is a maximal rec. support

Back

Find Maximal Recoverable Index Set



Special Cases:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & -2 \end{pmatrix}$$

- Support $I_4 = [-1, 0, 1]$ is a maximal rec. support

Back

Consider $x = (x_1, ..., x_n)^T \in \mathbb{R}^n, p \in \mathbb{N} \setminus \{0\}$

$$\|x\|_p^p := \sum_{i=1}^{n} |x_i|^p.$$

Example:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|,$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

Back