Inverse problems in spaces of measures

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Outline

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Inverse problems and sparsity

Linear inverse problems

Solve $Ku = f$

Available $f^\delta = f + \eta^\delta$

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Topics

- Hilbert/Banach space setting
  $\rightsquigarrow$ Ill-posedness
- Tikhonov regularization
- Convergence (rates) as $\delta \to 0$
- Numerical minimization algorithms

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Compressed sensing

Find $\|u\|_0^0 \to \min$

with $Ku = f$

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Topics

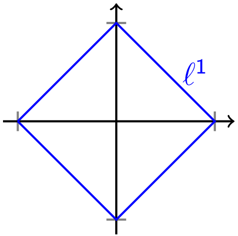
- Finite-dimensional setting
- Combinatorical problem
  $\rightsquigarrow$ NP-hard
- $\|u\|_1$ instead of $\|u\|_0^0$: with high probability same results for K random matrix

Inverse problems and sparsity

Inverse problems with sparsity constraints

$$\min_{u \in \ell^1} \frac{\|KBu - f^\delta\|_{H_2}^2}{2} + \alpha \|u\|_1$$

- $K : H_1 \to H_2$ forward model
- $B : \ell^1 \to H_1$ basis synthesis



- sparsity w.r.t. a fixed basis is enforced by the ℓ^1-norm
- ℓ^1 reflects that the basis is discrete

$\rightsquigarrow$ continuous setting?

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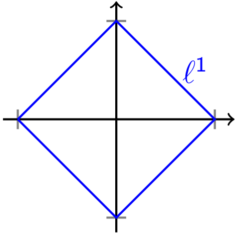

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Replace ℓ^1 by $\mathcal{M}(\Omega)$, the space of Radon measures

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$$\min_{u \in \mathcal{M}(\Omega)} \frac{\|A^* u - f^\delta\|_{H_2}^2}{2} + \alpha \|u\|_{\mathcal{M}}$$

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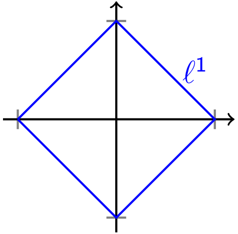
$\rightsquigarrow$ $A^* : \mathcal{M}(\Omega) \to H_2$ combined model



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Tikhonov functional in Radon space

Tikhonov regularization

$$u_\alpha^\delta \in \operatorname*{arg\,min}_{u \in \mathcal{M}(\Omega)} \frac{\|A^*u - f^\delta\|_H^2}{2} + \alpha \|u\|_{\mathcal{M}}$$

Setting

- H Hilbert space
- Ω sep. locally compact space
- $\mathcal{M}(\Omega) = \mathcal{C}_0(\Omega)^*$
  space of signed Radon measures
- $A \in \mathcal{L}\big(H, \mathcal{C}_0(\Omega)\big)$
  predual forward model
  (B weak*-cont. $\Leftrightarrow B = A^*$)

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Theorem

There exists a minimizer.

Proof:
Direct method for weak*-convergence in $\mathcal{M}(\Omega)$ □

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- Case $\Omega \subset \mathbb{R}^d$ open $\rightsquigarrow$ *[Scherzer/Walch '08]*

Optimality conditions

Original problem $(\mathcal{P})$

$$\min_{u \in \mathcal{M}(\Omega)} \frac{\|A^* u - f^\delta\|_H^2}{2} + \alpha \|u\|_{\mathcal{M}}$$

Predual problem $(\mathcal{P}_0)$

$$\min_{v \in H} \frac{\|v - f^\delta\|_H^2}{2} + I_{\{\|Av\|_\infty \le \alpha\}}(v)$$

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Proof Set
$$\begin{cases} S_0(v) = \frac{1}{2}\|v - f^\delta\|_H^2 & \text{continuous} \\ R_0(\mu) = I_{\{\|\mu\|_\infty \leq \alpha\}}(\mu) & \text{continuous at } 0 \end{cases}$$

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Fenchel-Rockafellar duality

$$\min_{v \in H} S_0(v) + R_0(Av) = \max_{u \in \mathcal{C}_0(\Omega)^*} -S_0^*(A^* u) - R_0^*(-u) \qquad (\mathcal{P}_0^*)$$

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Fenchel conjugates

$$\left.\begin{aligned} S_0^*(w) &= \tfrac{1}{2}\|w+f^\delta\|_H^2 - \tfrac{1}{2}\|f^\delta\|_H^2 \\ R_0^*(u) &= \alpha\|u\|_{\mathcal{M}} \end{aligned}\right\} \xrightarrow[u\to -u]{\max -E = -\min E} (\mathcal{P}_0^*) \sim (\mathcal{P})$$

□

Optimality conditions

Primal-dual system

$$\left.\begin{aligned} -A^* u^* &\in \partial S_0(v^*) \\ u^* &\in \partial R_0(Av^*) \end{aligned}\right\} \xLeftrightarrow{\partial S_0(v) = \{v - f^\delta\}} u^* \in \partial R_0\bigl(A(A^* u^* - f^\delta)\bigr)$$

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- Jordan-Hahn decomposition of u^* + normal cone of constraints R_0

Optimality system

$$\begin{cases} \|A(A^* u^* - f^\delta)\|_\infty \leq \alpha \\ \operatorname{supp} u^* \subset \{|A(A^* u^* - f^\delta)| = \alpha\} \\ u^* \leq 0 \text{ on } \{A(A^* u^* - f^\delta) = \alpha\} \\ u^* \geq 0 \text{ on } \{A(A^* u^* - f^\delta) = -\alpha\} \end{cases}$$

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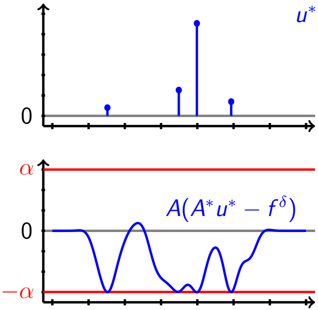

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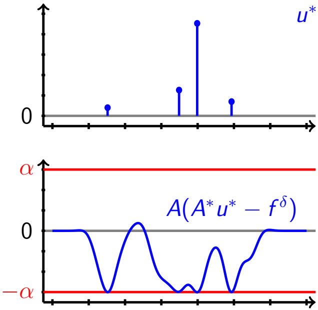

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$\rightsquigarrow$ **Sparse solutions possible**

Stability and regularization properties

Standard result

- $u^\dagger$ unique minimum-norm solution, $f^\dagger = A^* u^\dagger$
- $\|f^\dagger - f^\delta\|_H \leq \delta$, $\dfrac{\delta^2}{\alpha} \to 0$

Then: $u_\alpha^\delta \rightharpoonup^* u^\dagger$.

Remark

Minimum-norm solution $u^\dagger$ non-unique

$\Rightarrow$ $u_\alpha^\delta \rightharpoonup^* u^*$ subsequentially

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Convergence rate

- There exists a $h \in H$ with

$\langle u^\dagger, Ah \rangle = \|u^\dagger\|_{\mathcal{M}}$, $\|Ah\|_\infty = 1$

- Parameter choice $\alpha \sim \delta$

Then: $D(u_\alpha^\delta, u^\dagger) = \mathcal{O}(\delta)$

- D Bregman distance w.r.t. Radon-norm and Ah
- Bases on *[Burger/Osher '04]*

Remark

In general weaker than weak*-convergence

Example: Sparse deconvolution

Minimization problem

$$\min_{u \in \mathcal{M}(\Omega - \Omega')} \frac{1}{2} \int_\Omega |u * k - f^\delta|^2 \, dx + \alpha \|u\|_{\mathcal{M}}$$

- $k \in L^2(\Omega')$
- $f^\delta \in L^2(\Omega)$

Predual operator

$$Aw = w * \bar{k}, \quad \bar{k}(x) = k(-x)$$

- $A : L^2(\Omega) \to \mathcal{C}_0(\Omega - \Omega')$ linear and continuous
- $A^* : \mathcal{M}(\Omega - \Omega') \to L^2(\Omega)$ convolution + restriction

Applications

- Finding peaks/isotopes in noisy mass-spectrometry data
- Detection of stars in ground-based telescope images

Example: Elliptic PDEs

Minimization problem

$$\min_{u \in \mathcal{M}(\Omega)} \frac{1}{2} \int_\Omega |L^{-1}u - f^\delta|^2 \, dx + \alpha \|u\|_{\mathcal{M}}$$

- $Ly = -\operatorname{div}(K \nabla y)$ uniformly elliptic
- $f^\delta \in L^2(\Omega)$

Predual operator

Aw solution to

$$\begin{cases} -\operatorname{div}(K\nabla y) = w & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases}$$

- $A : L^2(\Omega) \to H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow \mathcal{C}_0(\Omega)$ if $\Omega \subset \mathbb{R}^d$, $d \leq 3$
- $A^* : \mathcal{M}(\Omega) \to L^2(\Omega)$ adjoint solution operator
- $A^* u = L^{-1} u$ solution to

$$\begin{cases} -\operatorname{div}(K\nabla y) = u & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases}$$

Application

- Optimal sensor/actuator placement

$\rightsquigarrow$ *[Stadler '09]*
[Clason/Kunisch '10]

Outline

Numerical minimization

Aim

- Produce sparse iterates $\rightsquigarrow u^n = \sum_i \mu_i^n \delta_{x_i^n}$

Algorithm

1. Set $u^0 = 0$, $M_0 = \|f^\delta\|_H^2/(2\alpha)$
2. $\begin{cases} \text{Compute } w^n = -A(A^*u^n - f^\delta) \\ \text{Find maximizer } x^* \text{ of } |w^n| \end{cases}$
3. Set $v^n = \begin{cases} \alpha^{-1} M_0 w^n(x^*) \delta_{x^*} & \text{if } |w^n(x^*)| > \alpha \\ 0 & \text{else} \end{cases}$
4. Compute $\begin{cases} u^{n+1/2} &= u^n + s_n(v^n - u^n) \\ s_n &= s(u^n, v^n) \in [0,1] \end{cases}$
5. Perform soft-thresholding step on the coefficients of $u^{n+1/2}$ analogous to *[DDD '04]* $\rightsquigarrow$ next iterate u^{n+1}

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$\rightsquigarrow$ **Successive peak insertion and thresholding**

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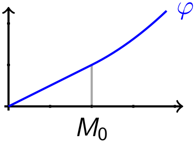
$\rightsquigarrow$ **Successive peak insertion and thresholding (SPInaT)**

Analysis of the algorithm

Equivalence

- Minimizers of $(\mathcal{P})$ coincide with minimizers of

$$\min_{u \in \mathcal{M}(\Omega)} \frac{\|A^* u - f^\delta\|_H^2}{2} + \varphi(\|u\|_{\mathcal{M}}) \qquad (\tilde{\mathcal{P}})$$



$$\varphi(t) = \begin{cases} \alpha t & \text{for } t \leq M_0 \\ \frac{\alpha}{2M_0}(t^2 + M_0^2) & \text{else} \end{cases}$$

if $M_0 \geq \|f^\delta\|_H^2/(2\alpha)$

- Functional $\Psi(u) = \varphi(\|u\|_{\mathcal{M}})$ is strongly coercive
  $\rightsquigarrow \partial\Psi : \mathcal{M}(\Omega) \rightrightarrows \mathcal{C}_0(\Omega)$ is onto

Analysis of the algorithm

Peak insertion (Steps 2–4)

- Amounts to generalized conditional gradient method on $(\tilde{\mathcal{P}})$

$$
\begin{cases}
w^n = A(A^* u^n - f^\delta) \\
v^n \in \underset{v \in \mathcal{M}(\Omega)}{\arg\min} \; \langle v, w^n \rangle + \Psi(v) \\
s_n = \min\Big\{1, \dfrac{\Psi(u^n) - \Psi(v^n) + \langle u^n - v^n, w^n \rangle}{\|A^*(v^n - u^n)\|_H^2}\Big\} \\
u^{n+1} = u^n + s_n(v^n - u^n)
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- Minimization step corresponds to $-w^n \in \partial\Psi(v^n)$
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- Minimization step corresponds to $-w^n \in \partial\Psi(v^n)$
  $\rightsquigarrow$ $-w^n \in \mathcal{C}_0(\Omega) \Rightarrow$ there exists such a v^n
- Explicit choice: $\begin{cases} x^* \in \Omega \text{ such that } |w^n(x^*)| = \|w^n\|_\infty \\ v^n = \alpha^{-1} M_0 w^n(x^*) \chi_{\{\|w^n\|_\infty > \alpha\}} \delta_{x^*} \end{cases}$

Analysis of the algorithm

Generalized conditional gradient method

- Consider functional distance

$$r(u) = \frac{1}{2}\|A^*u - f^\delta\|_H^2 + \Psi(u) - \min_{\bar{u}\in\mathcal{M}(\Omega)} \left\{ \frac{1}{2}\|A^*\bar{u} - f^\delta\|_H^2 + \Psi(\bar{u}) \right\}$$

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Theorem

(GCGM) generates a sequence $\{u^n\}$ such that

$$r(u^{n+1}) \leq r(u^n) \qquad \text{and} \qquad r(u^n) = \mathcal{O}(n^{-1})$$

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- Proof bases on techniques in *[Dunn '79]* and *[B./Lorenz '08]*

Sketch of proof

- v^n solution of "linearized" problem + choice of s_n $\Big\} \Rightarrow$ $r(u^{n+1}) - r(u^n) \leq -c(u^n)r(u^n)^2$
- Strong coercivity of $\Psi \Rightarrow c(u^n) > c > 0$
- Summation trick $\rightsquigarrow r(u^n) = \mathcal{O}(n^{-1})$ □

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Theorem

(GCGM) generates a sequence $\{u^n\}$ such that

$$r(u^{n+1}) \leq r(u^n) \qquad \text{and} \qquad r(u^n) = \mathcal{O}(n^{-1})$$

- Proof bases on techniques in *[Dunn '79]* and *[B./Lorenz '08]*

Sketch of proof

- $\left.\begin{array}{r} v^n \text{ solution of ``linearized'' problem} \\ + \text{ choice of } s_n \end{array}\right\} \Rightarrow \begin{array}{l} r(u^{n+1}) - r(u^n) \\ \quad \leq -c(u^n) r(u^n)^2 \end{array}$
- Strong coercivity of $\Psi \Rightarrow c(u^n) > c > 0$
- Summation trick $\rightsquigarrow r(u^n) = \mathcal{O}(n^{-1})$ □

Analysis of the algorithm

Generalized conditional gradient method

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Soft thresholding (Step 5)

- Suppose: $u = B\mu = \sum_{i=1}^{N} \mu_i \delta_{x_i}$ with peak positions x_i fixed $\rightsquigarrow$ $(\mathcal{P})$ reduces to

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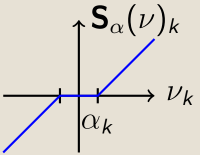
Numerical solution

- Iterative thresholding *[DDD '04]*

$$\mu^{n+1} = \mathbf{S}_{s\alpha}\big(\mu^n - sB^*A(A^*B\mu^n - f^\delta)\big)$$

- $\mathbf{S}_\alpha(\nu)_k = \operatorname{sgn}(\nu_k)\max(0, |\nu_k| - \alpha)$



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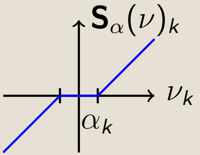

- In particular: $r(B\mu^{n+1}) \le r(B\mu^n)$ for $s > 0$ appropriate

Analysis of the algorithm

Combining the steps

- $u^n = \sum_i \mu_i^n \delta_{x_i^n}$
- Peak insertion: $\begin{cases} \text{Add at most one } \delta\text{-peak at } x_*^n \\ u^{n+1/2} = \sum_i \mu_i^{n+1/2} \delta_{x_i^n} + \mu_*^n \delta_{x_*^n} \end{cases}$
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Theorem *[B./Pikkarainen '10]*

- $r(u^{n+1}) \leq r(u^n)$ and $r(u^n) = \mathcal{O}(n^{-1})$
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- Properties of (GCGM) and soft thresholding:

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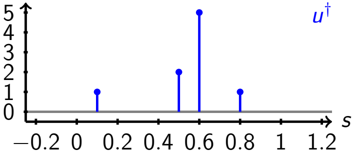
Outline

Sparse deconvolution



1D Deconvolution

$$\min_{u \in \mathcal{M}(\Omega)} \frac{1}{2}\|u * k - f^\delta\|_H^2 + \alpha\|\mu\|_{\mathcal{M}}$$

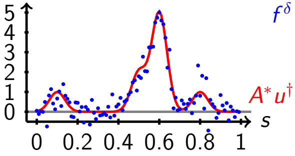
- H discrete space, k cubic B-spline

$\rightsquigarrow$ $\mathcal{M}(\Omega)$ **is not discretized**

Additional tweaks

- Merge peaks if functional value decreases
- Gradient flow w.r.t. peak positions

Sparse deconvolution



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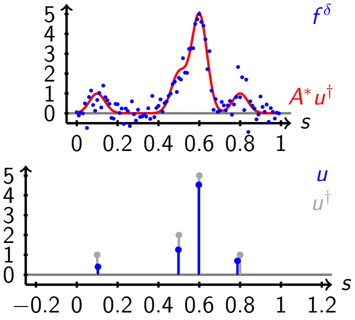

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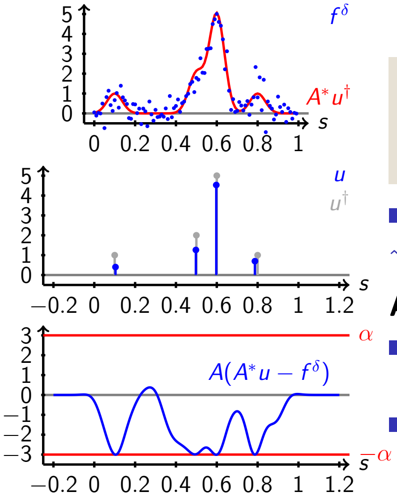

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Outline

Summary

- Framework for continuous sparsity in inverse problems $\rightsquigarrow$ Tikhonov minimization in measure space $\mathcal{M}(\Omega)$
- Enjoys well-posedness and regularization properties
- Applicable for sparse deconvolution and optimal sensor/actuator placement
- A flexible weak*-convergent iterative minimization algorithm can be developed
- Method may be realized without discretizing $\mathcal{M}(\Omega)$

More information

K. Bredies and H. Pikkarainen.
Inverse problems in spaces of measures.
ESAIM: Control, Optimisation and Calculus of Variations 19(1):190–218, 2013.