Exact Support Recovery for Sparse Spikes Deconvolution

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Deconvolution

Measuring devices have a non sharp impulse response: our observations are **blurred** of a ”true ideal scene”.

- Geophysics,
- Astronomy,
- Microscopy,
- Spectroscopy,
- . . .

Image courtesy of S. Ladjal

Goal: Obtain as much detail as we can from given measurements.

Outline

1. Mathematical model
2. Robustness for the continuous problem
3. Asymptotics for the discrete problems

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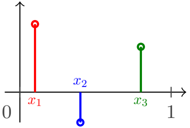
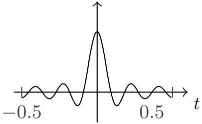
The Deconvolution Problem

Consider a signal m_0 defined on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (i.e. $[0,1)$ with periodic boundary condition).

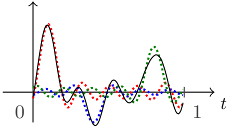
$m_0 \quad * \quad \varphi \quad = \quad y_0$



Original Signal



Low-pass filter



Observation

The Deconvolution Problem

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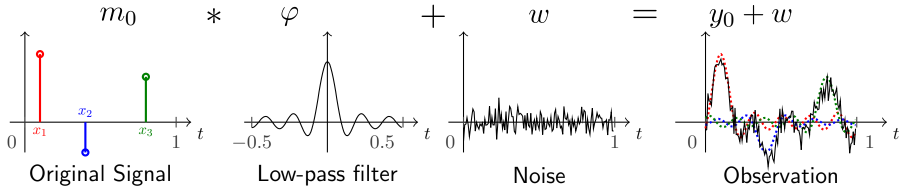

Recovery of m_0 = Ill-posed problem

The Deconvolution Problem

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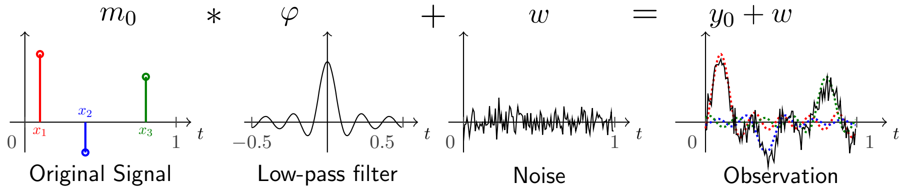

- Assumption on the signal: m_0 is a **sparse** Radon measure

$$m_0 = \sum_{i=1}^{N} a_{0,i}\delta_{x_{0,i}}, \quad \text{where} \begin{cases} a_{0,i} \in \mathbb{R}, \\ x_{0,i} \in \mathbb{T}, \\ N \in \mathbb{N} \text{ is small.} \end{cases}$$

so that we observe $y_0 + w = \sum_{i=1}^{N} a_{0,i}\varphi(\cdot - x_{0,i}) + w$.

- Idea: Look for a **sparse** signal m such that $\varphi * m \approx y_0$ (or $y_0 + w$).

Gridless Regularization

- Following (de Castro and Gamboa, 2012; Bredies and Pikkarainen, 2013; Candès and Fernandez-Granda, 2013b), we consider the continuous counterpart of the ℓ^1 norm, the **total variation of measures**:

$$\forall m \in \mathcal{M}(\mathbb{T}), \quad |m|(\mathbb{T}) = \sup \left\{ \int \psi \mathrm{d}m \; ; \; \psi \in C(\mathbb{T}), \|\psi\|_\infty \leqslant 1 \right\}.$$

  - if $dm = f(t)dt$, then $|m|(\mathbb{T}) = \int_{\mathbb{T}} |f(t)|dt$,
  - if $m = \sum_{i=1}^N a_i \delta_{x_i}$, then $|m|(\mathbb{T}) = \sum_{i=1}^N |a_i|$.

- Define $\Phi : \mathcal{M}(\mathbb{T}) \to \mathrm{L}^2(\mathbb{T})$ by

$$\Phi m : t \mapsto \int_{\mathbb{T}} \varphi(x - t) \mathrm{d}m(x).$$

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Beurling minimal extrapolation (de Castro and Gamboa, 2012)

$$\inf_{m \in \mathcal{M}(\mathbb{T})} |m|(\mathbb{T}) \text{ such that } \Phi m = y_0 \qquad (\mathcal{P}_0(y_0))$$

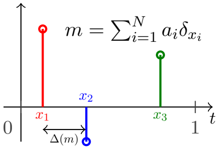
Beurling LASSO (Bredies and Pikkarainen, 2013; Azais et al., 2013)

$$\inf_{m \in \mathcal{M}(\mathbb{T})} \lambda |m|(\mathbb{T}) + \frac{1}{2} \|\Phi m - (y_0 + w)\|_2^2 \qquad (\mathcal{P}_\lambda(y_0 + w))$$

Identifiability for discrete measures

Minimum separation distance of a measure m:

$$\Delta(m) = \min_{x,x' \in \operatorname{Supp} m, x \neq x'} |x - x'|$$



Ideal Low Pass filter: $\varphi(t) = \frac{\sin(2f_c+1)\pi t)}{\sin \pi t}$
i.e $\hat{\varphi}_n = 1$ for $|n| \leqslant f_c$, 0 otherwise.

Theorem ((Candès and Fernandez-Granda, 2013b))

Let φ be the ideal low-pass filter. There exists a constant $C > 0$ such that, for any (discrete) measure m_0 with $\Delta(m_0) \geqslant \frac{C}{f_c}$, m_0 is the unique solution of

$$\inf_{m \in \mathcal{M}(\mathbb{T})} |m|(\mathbb{T}) \text{ such that } \Phi m = y \qquad (\mathcal{P}_0(y))$$

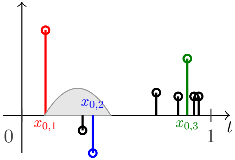
where $y = \Phi m_0$.

Remark: $1 \leqslant C \leqslant 1.87$.

Outline

Robustness of the continuous problem?

$m_0^* = \sum_{i=0}^N a_{0,i}^* \delta_{x_{0,i}^*}$



- (Bredies and Pikkarainen, 2013) the solutions to $\mathcal{P}_\lambda(y+w)$ converge to m_0 **for the weak-* topology** as $\lambda \to 0$, $\frac{\|w\|^2}{\lambda} \to 0$.
- (Azais et al., 2013): (ideal LPF) the spikes are in "boxes" around the original spikes

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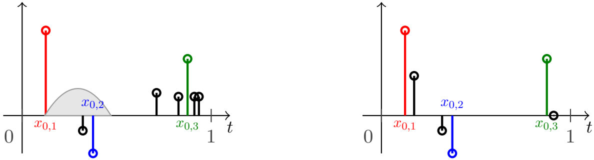

No Support recovery

"Support recovery"

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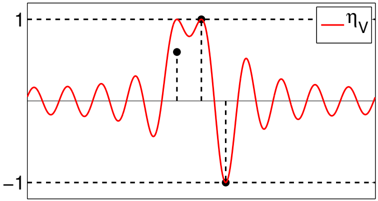
Subdifferential of the total variation

Subdifferential of the total variation

$$\begin{aligned}\partial|m|(\mathbb{T}) &= \left\{\eta \in C(\mathbb{T}); \quad \forall m' \in \mathcal{M}(\mathbb{T}),\ |m'|(\mathbb{T}) \geqslant |m|(\mathbb{T}) + \langle \eta, m' - m\rangle\right\}\\ &= \{\eta \in C(\mathbb{T});\ \|\eta\|_\infty \leqslant 1, \forall t \in \operatorname{Supp} m_+\ \eta(t) = 1,\\ &\qquad \text{and } \forall t \in \operatorname{Supp} m_-\ \eta(t) = -1\}\end{aligned}$$

Example: For $m = \sum_{i=1}^N a_i \delta_{x_i}$,
$\partial|m|(\mathbb{T}) = \{\eta \in C(\mathbb{T}); \|\eta\|_\infty \leqslant 1$ and $\eta(x_i) = \operatorname{sign}(a_i)$ for $1 \leqslant i \leqslant N\}$.



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$$= \{\eta \in C(\mathbb{T});\ \|\eta\|_\infty \leqslant 1, \forall t \in \operatorname{Supp} m_+\ \eta(t) = 1,$$
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Optimality conditions

- For $\mathcal{P}_0(y_0)$:

$$\exists \eta \in \partial|m_0|(\mathbb{T}) \text{ such that } \eta \in \operatorname{Im} \Phi^*.$$

- For $\mathcal{P}_\lambda(y_0)$:

$$\exists \eta_\lambda \in \partial|m_\lambda|(\mathbb{T}) \text{ such that } \lambda\eta_\lambda + \Phi^*(\Phi m_\lambda - y_0) = 0.$$

η (resp. η_λ) is a **certificate** for m_0 (resp. m_λ), and

Find the support of m_0 (resp. m_λ) $\rightsquigarrow$ Find all t such that $\eta(t) = \pm 1$ (resp. $\eta_\lambda(t) = \pm 1$)

Minimal norm certificate

Define the **minimal norm certificate** for m_0 as

$$\eta_0 = \Phi^* p_0, \quad \text{where} \quad p_0 = \operatorname{argmin}\{\|p\|_{L^2(\mathbb{T})}; \Phi^* p \in \partial|m_0|(\mathbb{T})\}.$$

Proposition (D.P.'2013)

Let η_λ be the certificate for $\mathcal{P}_\lambda(y_0)$. Then η_λ converges to the minimal norm certificate η_0 as $\lambda \to 0^+$,

$$\lim_{\lambda \to 0^+} \eta_\lambda^{(k)} = \eta_0^{(k)}.$$

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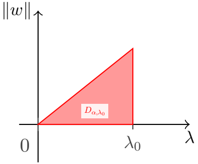
Definition (Non Degenerate Source Condition)

We say that $m_0 = \sum_{i=1}^N a_{0,i} x_{0,i}$ satisfies the **Non Degenerate Source Condition** (NDSC) if η_0 exists (= *source condition*) and

- for all $t \in \mathbb{T} \setminus \{x_{0,1}, \ldots x_{0,N}\}$, $\eta_0(t) < 1$,
- for all $i \in \{1, \ldots N\}$, $\eta_0''(x_{0,i}) \neq 0$.

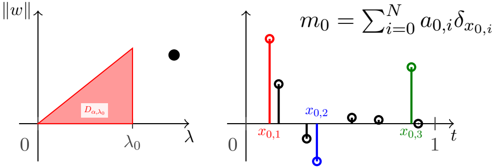
Robustness of the support



Define, for $m_0 = \sum_{i=1}^N a_{0,i}\delta_{x_{0,i}}$,

- the "matrix"
  $\Gamma_{x_0} = (\varphi(\cdot - x_{0,1}), \dots \varphi(\cdot - x_{0,N}), \varphi'(\cdot - x_{0,1}), \dots \varphi'(\cdot - x_{0,N}))$,
- the set $D_{\alpha,\lambda_0} = \{(\lambda, w) \in \mathbb{R} \times L^2(\mathbb{T}), 0 \leqslant \lambda \leqslant \lambda_0 \text{ and } \|w\| \leqslant \alpha\lambda\}$.

Robustness of the support



Theorem ((D.-P. 2013))

Let $m_0 = \sum_{i=1}^N a_i \delta_{x_{0,i}}$ *such that* m_0 *satisfies the* **Non Degenerate Source Condition** *and that* Γ_{x_0} *has full rank. Then there exists,* $\alpha > 0$, $\lambda_0 > 0$ *such that on* D_{α,λ_0}.

- *the solution* $\tilde{m}_\lambda$ *to* $\mathcal{P}_\lambda(y+w)$ *is unique and has exactly* N *spikes,* $\tilde{m}_\lambda = \sum_{i=1}^N \tilde{a}_{\lambda,i}\delta_{\tilde{x}_{\lambda,i}}$,
- *the mapping* $(\lambda, w) \mapsto (\tilde{a}_\lambda, \tilde{x}_\lambda)$ *is* C^1.

In particular, for $\lambda = \frac{1}{\alpha}\|w\|$, $|x_{0,i} - \tilde{x}_{\lambda,i}| = \mathcal{O}(\|w\|)$ and $|a_{0,i} - \tilde{a}_{\lambda,i}| = \mathcal{O}(\|w\|)$ for $1 \leqslant i \leqslant N$.

How to check the Non Degenerate Source Condition?

Let $m_0 = \sum_{i=1}^{N} a_{i,0} x_{i,0}$, and define the vanishing derivatives precertificate:

$$\eta_V = \Phi^* \Gamma_x^{+,*} \begin{pmatrix} s \\ 0_N \end{pmatrix},$$

where $\quad s = (\mathrm{sign}(a_{0,i}))_{1 \leqslant i \leqslant N} \quad$ and $\quad 0_N = (0, \ldots, 0)^T \in \mathbb{R}^N.$

Proposition (Non Degenerate Source Condition)

m_0 *satisfies the* **Non Degenerate Source Condition** *(NDSC) if and only if*

- *for all* $t \in \mathbb{T} \setminus \{x_{0,1}, \ldots x_{0,N}\}$, $\eta_V(t) < 1$,
- *for all* $i \in \{1, \ldots N\}$, $\eta_V''(x_{0,i}) \neq 0$.

In that case $\eta_0 = \eta_V$.

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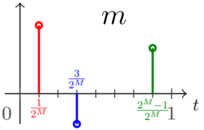
Discretization

Define a finite grid $\mathcal{G}_M = \{\frac{k}{2^M};\ 0 \leqslant k \leqslant 2^M - 1\} \subset \mathbb{T}$, and impose that the support of the solution is on the grid

$$m = \sum_{k=0}^{2^M-1} b_k \delta_{\frac{k}{2^M}}.$$



Candidate Signal

- **Basis Pursuit** (Chen and Donoho, 1994)

$$\inf_{m \in \mathcal{M}(\mathbb{T})} |m|(\mathbb{T}) \text{ such that } \Phi m = y_0 \quad \text{and} \quad \operatorname{Supp} m \subset \mathcal{G}_M \qquad (\mathcal{P}_0^M(y_0))$$

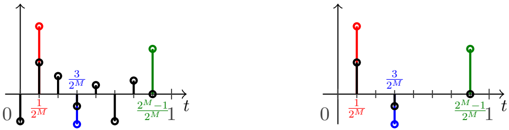
- **LASSO** (Tibshirani, 1996) or **Basis Pursuit Denoising** (Chen et al., 1999)

$$\inf_{m \in \mathcal{M}(\mathbb{T})} \lambda |m|(\mathbb{T}) + \frac{1}{2}\|\Phi m - (y_0 + w)\|_2^2 \quad \text{and} \quad \operatorname{Supp} m \subset \mathcal{G}_M \qquad (\mathcal{P}_\lambda^M(y_0 + w))$$

Robustness

- ℓ^2-robustness (Grasmair et al. (2011))
- Support recovery
  - **Exact Recovery Principle (ERC)** (Tropp, 2006)
  - **Weak Exact Recovery Principle (W-ERC)** (Dossal and Mallat, 2005)
  - **Fuchs criterion** (Fuchs, 2004)

$m_0^* = \sum_{i=0}^{2^M-1} a_{0,i}^* \delta_{i/2^M}$



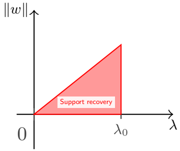
No support recovery | Support recovery

Fuchs theorem

For $m_0 = \sum_{i=1}^{N} a_{i_0} \delta_{x_{i,0}}$, define

$$\eta_F = \Phi^* p_F, \text{ where}$$

$$p_F = \operatorname{argmin}\{\|p\|_{L^2(\mathbb{T})}; (\Phi^* p)(x_{0,i}) = \operatorname{sign}(a_{i,0})\}$$
$$= \Phi^{+,*} s.$$



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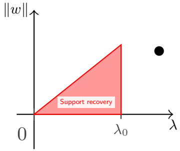
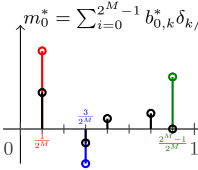




Theorem (Fuchs (2004))

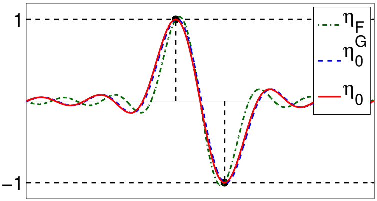
Assume that $\{\varphi(\cdot - x_{0,1}), \dots \varphi(\cdot - x_{0,N})\}$ *has full rank.*
If $|\eta_F(\frac{k}{2^M})| < 1$ *for all* k *such that* $\frac{k}{2^M} \notin \{x_{0,1}, \dots, x_{0,N}\}$, *then* m_0 *is the unique solution to* $\mathcal{P}_0^M(y)$, *and there exists* $\alpha > 0$, $\lambda_0 > 0$ *such that for* $0 \leqslant \lambda \leqslant \lambda_0$ *and* $\|w\|_2 \leqslant \alpha\lambda$,

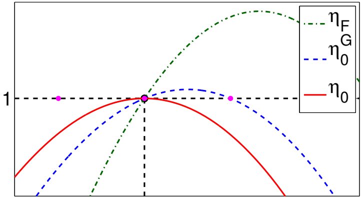
- *The solution* m_λ^M *to* $\mathcal{P}_\lambda^M(y + w)$ *is unique.*
- $\operatorname{Supp} m^M = \operatorname{Supp} m_0$, *that is* $m_\lambda^M = \sum_{i=1}^{N} a_{\lambda,i}^M \delta_{x_{0,i}}$, *and* $(\operatorname{sign} a_{\lambda,i}^*) = \operatorname{sign} a_{0,i})$,
- $a_{\lambda,I}^M = a_{0,I} + \Phi_{x_I}^+ w - \lambda(\Phi_{x_I}^* \Phi_{x_I})^{-1} \operatorname{sign}(a_{0,I})$.

If $|\eta_F(\frac{k}{2^M})| > 1$ *for some* k, *the support is not stable.*

Fuchs criterion is not valid for thin grids

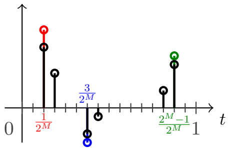
- η_F interpolates the sign
- Fuchs criterion: $|\eta_F(\frac{k}{2^M})| < 1$ for all k such that $\frac{k}{2^M} \notin \{x_{0,1}, \ldots, x_{0,N}\}$





In general, the support **cannot be stable** for thin grids, but...

Almost-robustness of the support



Theorem ((D.-P. 2013))

Let $m_0 = \sum_{0=1}^N a_{0,i}\delta_{x_{0,i}}$ *be a discrete measure, such that* $x_{0,i} \in \{\frac{k}{2^{M_0}}, 0 \leqslant k \leqslant 2^{M_0}-1\}$. *Assume that* m_0 *satisfies the* **Non Degenerate Source Condition**. *Then, for* M *large enough, there exists,* $\alpha^M > 0$, $\lambda_0^M > 0$ *such that for* $(\lambda, w) \in D_{\alpha^M,\lambda_0^M}$, *for all* m_λ^N *solution of* $\mathcal{P}_\lambda^N(y_0+w)$,

$$\operatorname{Supp} m_\lambda^N \subset \bigcup_{i=1}^N \{x_{0,i}, x_{0,i} + \frac{\varepsilon}{2^M}\} \text{ where } \varepsilon_i \in \{-1, 0, +1\}.$$

The support for small noise is made at most of **pairs of consecutive spikes**: the original one and one of its immediate neighbours.

Conclusion

- Support recovery for the continuous problem.
- Asymptotics of the discrete problems: "almost robustness" of the support
- Experiment yourself the Sparse Spikes Deconvolution on Numerical tours!

www.numerical-tours.com

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