Sparse and Cosparse Tomographic Recovery from Few Projections

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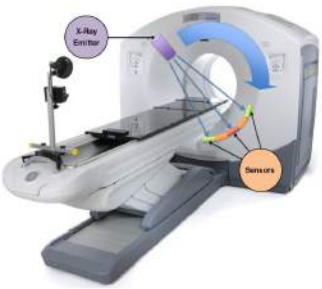
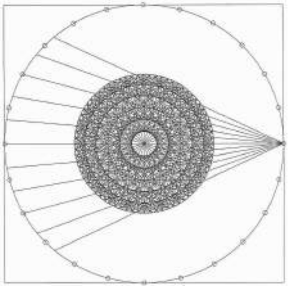
Standard Tomography, Many Projections

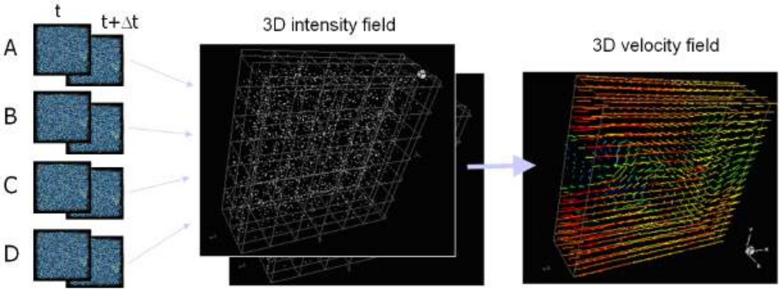




This talk is not about computerized tomography from a full set of projections.

Non-Standard Tomography, Few Projections

Tomographic Particle Image Velocimetry (Tomo PIV)



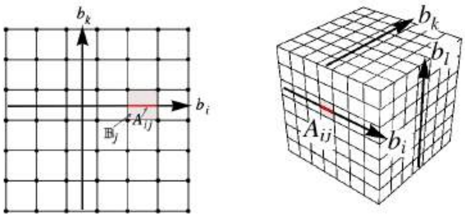
Particles in 3D are reconstructed from **3-6 projections** (2D images).

Non-Standard Tomography, Few Projections

Non-destructive testing (NDT) for 3D quality inspection

Compound bodies in 3D are reconstructed from up to **9 projections**.

Tomographic Imaging Set-Up

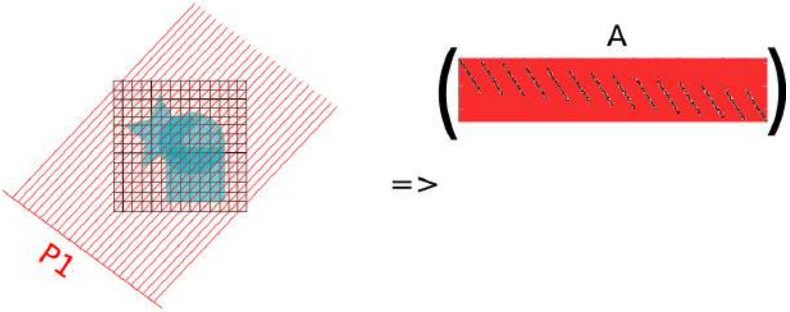


After discretization one obtains a linear system

$$Ax = b.$$

- Unknown: $x \in \mathbb{R}^n$ is the discretized image
- Measurements: $b \in \mathbb{R}^m$, with $b_i = \int_{L_i} x_j \mathbb{B}_j(z) dz$
- **Highly underdetermined** projection matrix $A \in \mathbb{R}^{m \times n}$, $m \ll n$, encodes the imagining geometry

Tomographic Sampling From Few Projections



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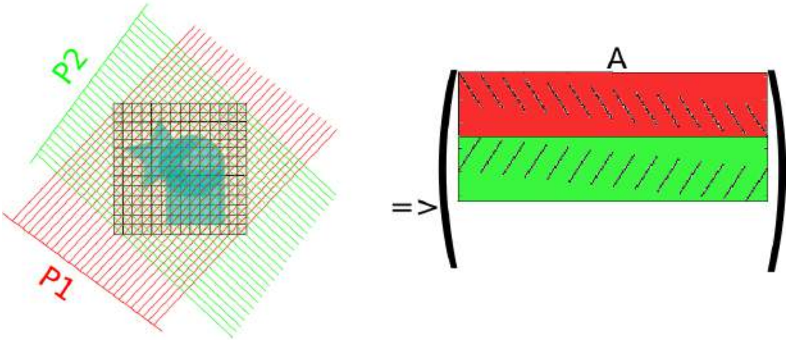

Simplification: Each pixel/voxel is intersected by the same number of rays $= \nu$

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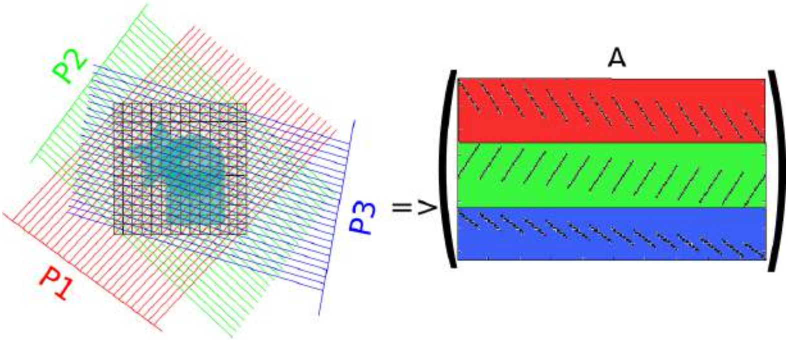

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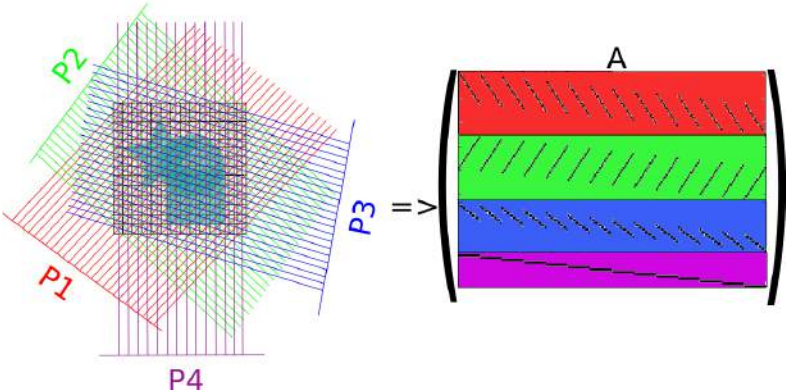

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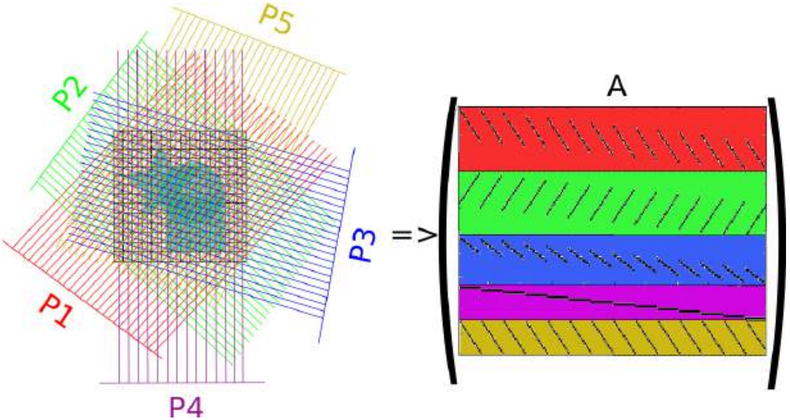
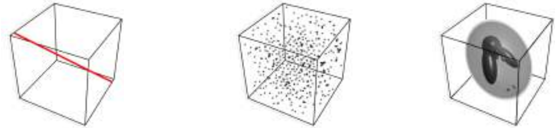


Simplification: Each pixel/voxel is intersected by the same number of rays $= \nu$

Recovery of "Simple" Signals?

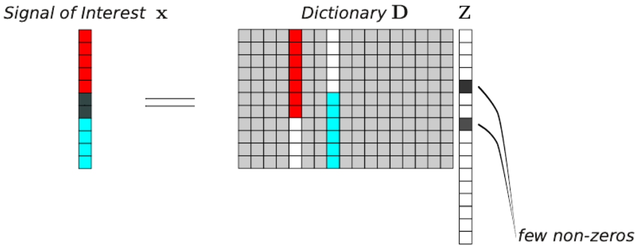
Rays: $m = 1.3 \cdot 10^5$ Sparse signal Cosparse signal



Can $n = 128^3 \approx 2 \cdot 10^6$ voxels be reconstructed exactly from $m = 1.3 \cdot 10^5$ line integrals corr. to 4 projections?

- Particle distributions are sparse.
- The gradient of the Shepp-Logan phantom is sparse.
- Benefits from the Compressed Sampling paradigm?

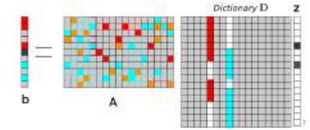
Sparsity Model I - Synthesis Model



Signal of interest is well approximated by a sum of few elements of a dictionary.

- $x \in \mathbb{R}^n$ is said k-sparse w.r.t D if z has a small number k of non-zero coefficients.
- **Particle images** are k-**sparse** w.r.t $D = I$.

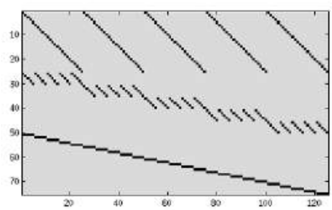
Compressed Sensing Likes Synthesis

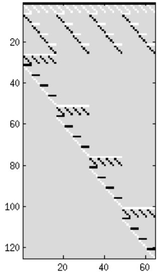


- Significantly fewer measurements $b \in \mathbb{R}^m$ suffice to reconstruct $x \in \mathbb{R}^n$ *exactly*, if
  - x is k-sparse w.r.t. D
  - AD satisfies certain properties (incoherency, nullspace property, RIP)
- ℓ_0/ℓ_1-equivalence
- Known sparsity trade-off for Gaussian matrices. Recovery of
  - all k-sparse signals if $k \leq \frac{m}{2e\log\left(\frac{n}{m}\right)}$ (worst case)
  - most k-sparse signals if $k \leq \frac{m}{2\log\left(\frac{n}{m}\right)}$ (average case)

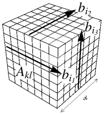
Tomographic Sensors are **POOR** CS Sensors

Unfortunately sensor matrices A have a highly sparse nullspace vectors!



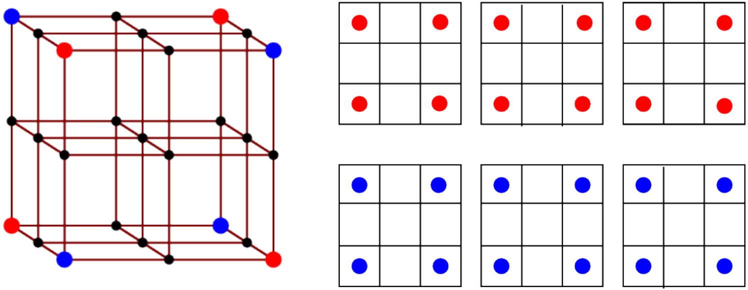


A $\ker(A)$

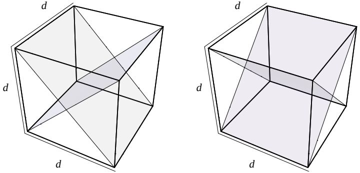


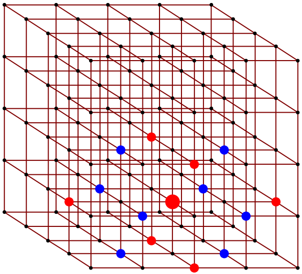
Tomo PIV: Poor Worst Case Performance

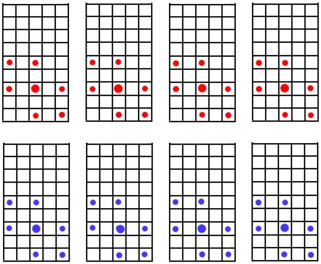
Non-unique particle configuration from 3 orthogonal views



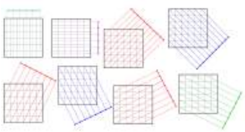
Other Geometries: 3D, 4 Views





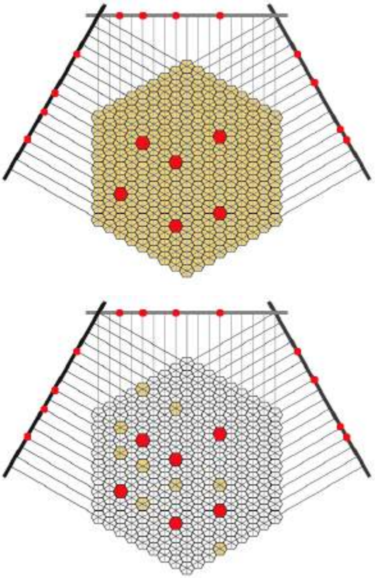


CS Properties of Tomographic Sensors - Other Geometries



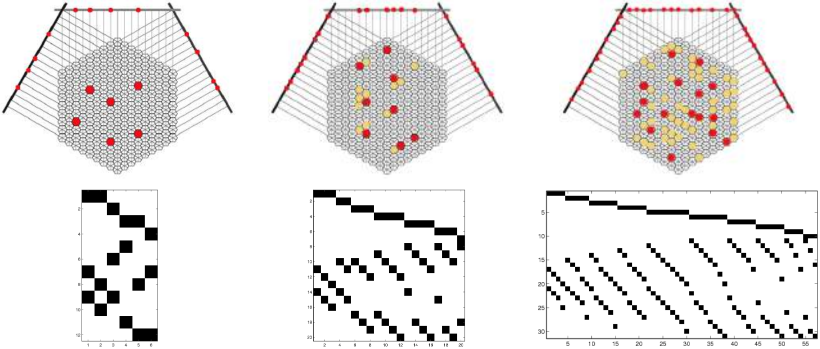
#views	n	m	max. k (worst case)
3		$4d-1$	2
4		$6d-2$	3
5	d^2	$7d+\lfloor\frac{d}{2}\rfloor-2$	5
6		$8d+2\lfloor\frac{d}{2}\rfloor-2$	7
7		$9d+3\lfloor\frac{d}{2}\rfloor-2$	7
8		$10d+4\lfloor\frac{d}{2}\rfloor-2$	7
3	d^3	$3d^2-3d+1$	3
4		$8d^2-4d$	6

Key Observation: Remove Redundant Cells



Average Performance Analysis of Reduced Systems?

- k uniformly distributed particles
- R_k = set of non-zero measurements
- C_k = set of non-redundant-cells



What about the recovery performance of $A_{R_k C_k} x = b_{R_k}$?

Average Case Analysis: Critical Sparsity Values

Compute $m_r(k) = \mathbb{E}[|R_k|]$ and $n_r(k) = \mathbb{E}[|C_k|]$ induced by random k particles.

Average Case Analysis: Critical Sparsity Values

Compute $m_r(k) = \mathbb{E}[|R_k|]$ and $n_r(k) = \mathbb{E}[|C_k|]$ induced by random k particles. Solve

$$\frac{m_r(k_\epsilon)}{n_r(k_\epsilon)} = \epsilon\nu, \quad \frac{m_r(k_{crit})}{n_r(k_{crit})} = 1, \quad \frac{m_r(k_{max})}{n_r(k_{max})} = \frac{1+\epsilon}{\nu},$$

with $\epsilon = \frac{\sqrt{5}-1}{2}$ and $\nu = \#\text{views}$.

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Unperturbed Systems

$Ax = b$ admits unique recovery of k-sparse nonnegative vectors x with high probability, if $k \leq \epsilon n_r(k_\epsilon) \approx k_\epsilon$.

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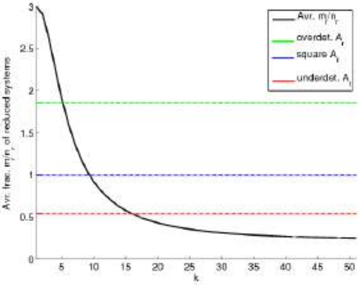
Perturbed Systems

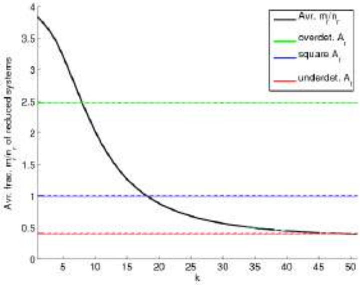
$\tilde{A}x = \tilde{b}$ admits unique recovery of k-sparse nonnegative vectors x with high probability, if $k \leq k_{crit}$. In case of complete Kruskal rank (of reduced systems), uniqueness holds if $k \leq \epsilon m_r(k_{max}) \approx k_{max}$.

Critical Sparsity Values and Recovery - Example: Hexagon

3 cameras



4 cameras



Bounding the Deviation of $m_r(k)$ and $n_r(k)$

Tail Bound

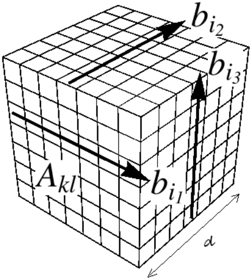
Let $m_r(k) = \mathbb{E}[|R_k|]$ and $m_r(k) = \mathbb{E}[|C_k|]$ for a given sparsity parameter k. Then, for any $\tau > 0$,

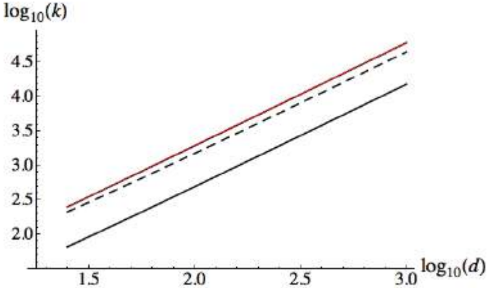
$$\Pr\left(||R_k| - m_r(k)| \geq \tau\right) \nearrow 2\exp\left(-\frac{\tau^2}{18k}\right) \qquad \text{if} \quad d \to \infty.$$

$$\Pr\left(\big||C_k| - n_r(k)\big| \geq \tau\right) \leq 2\exp\left(-\frac{\tau^2}{2p_0\hat{r}^2\left(1 + \frac{\tau p_1 d}{3p_0\hat{r}^2}\right)}\right),$$

with $p_0 \approx kd^{-2}$, $\hat{r}^2 \approx 3k^2 + 3kd\left(1 - \frac{k^2}{d^4}\right)$ and $p_1 \approx 1 - kd^{-2}$.

Theoretical Trade-Off

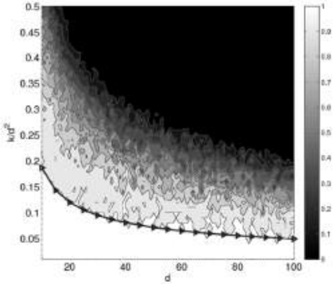




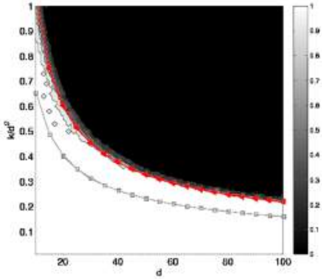
k_{max}, k_{crit}, k_ϵ (top to bottom) exibit a power law.
E.g. $k_{max}(d) \approx 3.54 d^{1.34}$

Theoretical and Empirical Trade-Off - 3D, 3 Views

unpert. A

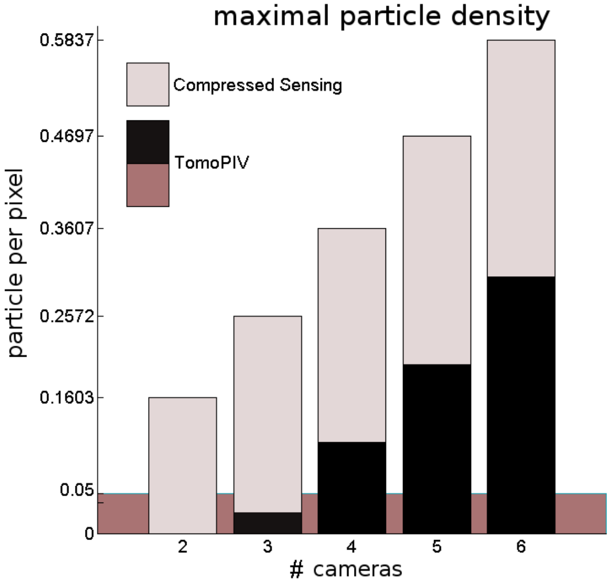


pert. $\tilde{A}$



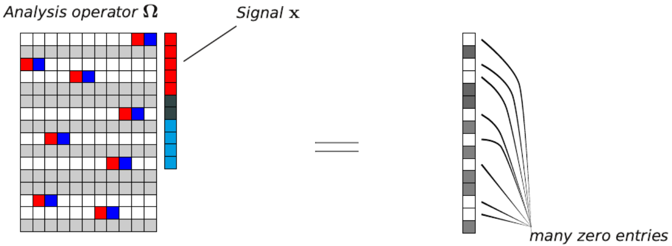
The empirical probability of a unique nonnegative solution accurately follows our theoretical prediction.

Average Case Sparsity Trade-Off: Gaussian vs. Tomo PIV



Recovery of Compound Bodies?

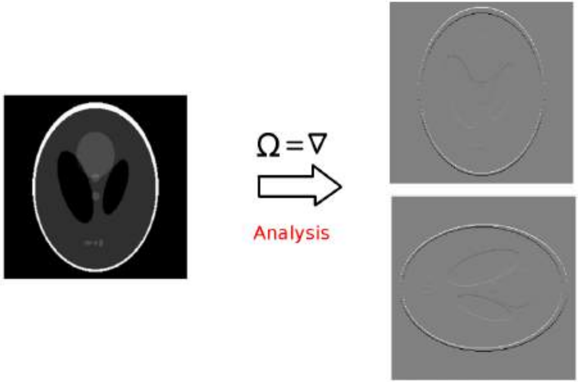
Sparsity Model II - Analysis Model



The signal is orthogonal to most of the rows of matrix Ω

- $x \in \mathbb{R}^n$ is said ℓ-cosparse, if for some *analysis operator* Ω
  - Ωx has a large number ℓ of zero coefficients.
- $\Lambda = \{i : (\Omega x)_i = 0\}$ is called cosupport of x.

Analysis Operator for Compound Objects

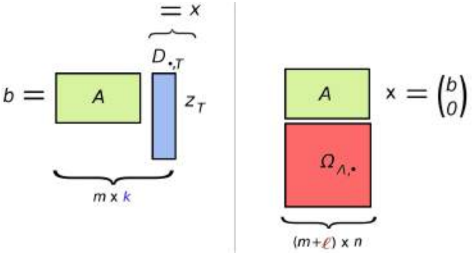


Known Support vs. Known Cosupport

Consider solving for x

$$b = Ax, \quad A \in \mathbb{R}^{m \times n}, \quad m \leq n.$$

and knowing T or Λ.



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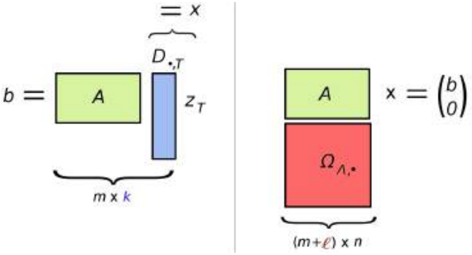


The zeros of the analysis representation carry the crucial information!

Uniqueness of a ℓ-Cosparse Signal

Sufficiency [Nam et al. '13]

Assume x^* is ℓ-cosparse and the rows of $\begin{pmatrix} A \\ \Omega \end{pmatrix}$ are lin. independent.

The system $Ax = Ax^*$ has at unique ℓ-cosparse solution if

$$\kappa_\Omega(\ell) := \max_{|\Lambda|=\ell} \dim \ker\{\Omega_\Lambda\} \leq \frac{m}{2}.$$

Number of Required Measurements

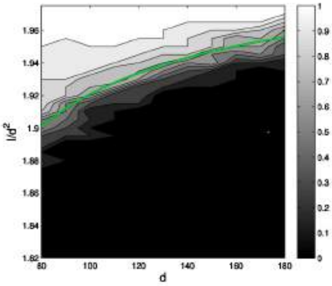
Key idea: evaluate $\kappa_\Omega(\ell)$ wrt. ∇ in 2D and 3D.

2D

$$m \geq 2n - (\ell + \sqrt{2\ell + 1} - 1)$$

3D

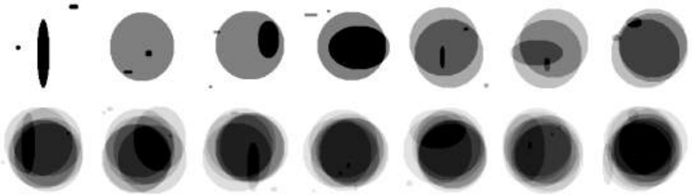
$$m \geq 2n - \frac{2}{3}\left(\ell + \sqrt[3]{3\ell^2} + 2\sqrt[3]{\frac{\ell}{3}} - 2\right)$$

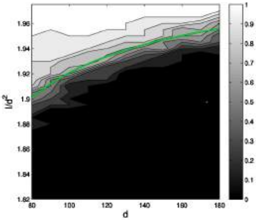


We solve (as a linear program)

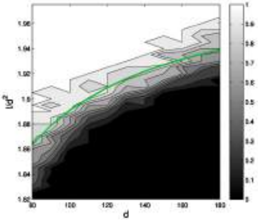
$$\min \|\nabla x\|_1 \quad \text{s.t.} \quad Ax = b, x \geq 0.$$

Recovery of Random Cosparse Images in 2D



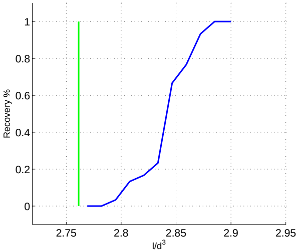


4 views

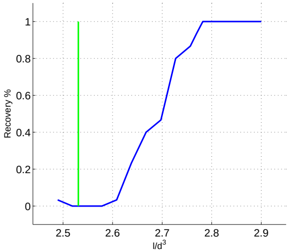


6 views

Recovery of Random Cosparse Images in 3D



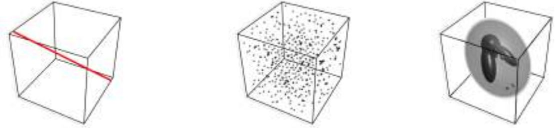
3 views



4 views

Conclusion: Exact Recovery for "Simple" Signals!

Rays: $m = 1.3 \cdot 10^5$ Sparse signal Cosparse signal



Exact reconstruction (with high probability) of $n = 128^3 \approx 2 \cdot 10^6$ voxels from $m = 1.3 \cdot 10^5$ line integrals corr. to 4 projections if

- particle distributions are $\leq 0.4\%$ sparse. Up to 8190 particles can be located exactly.
- the compound object is $\geq 95\%$ cosparse. The Shepp-Logan phantom is recoverable.

Details...

Web `http://ipa.iwr.uni-heidelberg.de`

Preprints *Phase Transitions and Cosparse Tomographic Recovery of Compound Solid Bodies from Few Projections* A. Denițiu, S. Petra, Cl. Schnörr and Ch. Schnörr, `arxiv.org/abs/1311.0423`

Average Case Recovery Analysis of Tomographic Compressive Sensing S. Petra, and C. Schnörr, Linear Algebra Appl. (2014)

Critical Parameter Values and Reconstruction Properties of Discrete Tomography: Application to Experimental Fluid Dynamics S. Petra, C. Schnörr, and A. Schröder, Fundam. Inform., (2013)

TomoPIV meets Compressed Sensing S. Petra, and C. Schnörr, Pure Math. Appl., (2009)